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Theory and Methodology

Multiperson decision-making based on multiplicative preference relations

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Abstract

A multiperson decision-making problem, where the information about the alternatives provided by the experts can be presented by means of different preference representation structures (preference orderings, utility functions and multiplicative preference relations) is studied. Assuming the multiplicative preference relation as the uniform element of the preference representation, a multiplicative decision model based on fuzzy majority is presented to choose the best alternatives. In this decision model, several transformation functions are obtained to relate preference orderings and utility functions with multiplicative preference relations. The decision model uses the ordered weighted geometric operator to aggregate information and two choice degrees to rank the alternatives, quantifier guided dominance degree and quantifier guided non-dominance degree. The consistency of the model is analysed to prove that it acts coherently. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A multiperson decision-making problem (MPDM) is defined as a decision situation in which a solution alternative(s) to a given question has (have) to be chosen, based on the information given by different people or experts [2,5,7,12–14,19]. Normally, the information can be presented by any of these three preference structures [5,20]:

1. *As a preference ordering of the alternatives*. In this case the alternatives are ordered from best to worst, without any other additional information.
2. *As a utility function*. In this case an expert gives a real valuation (a physical or a monetary value) for each alternative, i.e., a function that associates each alternative with a real number. This indicates the performance of that alternative according to his point of view.

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3. *As a preference relation*. This is the most usual case because most procedures in decision-making problems are based on pair comparison, in the sense that processes are linked to some degree of credibility of preference of any alternative over another.

In many decision situations it is quite natural to think that different experts can provide their evaluations by means of different preference structures. In such situations, before achieving the solution alternative(s), we have to make the information uniform using only one of the preference structure. We propose to use preference relations as the base for uniform representation [3,5,6,9,10,17–19,23,24]. Once this is done, a selection process is applied in order to obtain the solution alternative(s). The most well-known selection models are:

1. *Fuzzy model.* In this case the base to uniform the information is the fuzzy preference relation and the solution alternative(s) is (are) obtained by means of choice functions or degrees [5,9–11,16,17].
2. *Multiplicative model.* In this case the base to uniform the information is the multiplicative preference relation and the solution alternative(s) is (are) obtained using the Analytic Hierarchy Process (AHP) [8,18,19,22,25,26].

In this paper, we study an MPDM problem that presents a higher degree of freedom for modelling preferences. We analyse how to solve MPDM problems where the evaluations are presented by any of these three preference structures, preference orderings, utility functions and multiplicative preference relations. In this paper, we do not concentrate on the evaluation of these preference structures, but on the study of a decision model which integrates these three preference structures using multiplicative preference relations as the base of the uniform representation. First of all, we obtain transformation functions from preference orderings and utility functions into multiplicative preference relations. The decision model is designed using the *fuzzy majority* concept [10]. This concept is applied by a *fuzzy linguistic quantifier* [28], which allows more flexibility in the decision model because all decisions are reached using a soft majority of people's preferences. Our decision model does not use the AHP to obtain the solution alternatives. It is built using a fuzzy majority guided aggregation operator, the Ordered Weighted Geometric (OWG) [6], and two fuzzy majority guided choice degrees defined for multiplicative preference relations, *quantifier guided dominance degree and quantifier guided non-dominance degree* [6]. Finally, we analyse the consistency of this multiplicative decision model to ensure that it acts coherently. We show that the transformation functions do not change the ranking of the alternatives established by the other representation structures, when we use the aforementioned choice degrees.

The paper is set out as follows. The MPDM problem under different preference structures is presented in Section 2. How to make the information uniform is discussed in Section 3. Section 4 deals with the decision model using multiplicative preference relation as the base for uniform representation. An example is given in Section 5 to illustrate the use of the decision model. Section 6 draws our conclusions. Appendix A contains the description of fuzzy majority and fuzzy linguistic quantifier concepts.

2. The MPDM problem

Let $X = \{x_1, x_2, \ldots, x_n\}$ $(n \geqslant 2)$ be a finite set of alternatives. These alternatives have to be classified from best to worst, using the information given by a set of experts, $E = \{e_1, e_2, \ldots, e_m\}$ $(m \geqslant 2)$. As each expert, $e_k \in E$, has their own ideas, attitudes, motivations and personality, it is quite natural to think that different experts will give their preferences in a different way. This leads us to assume that the experts' preferences over the set of alternatives, X, may be represented in one of the following three ways:

1. *A preference ordering of the alternatives.* In this case, an expert, e_k, gives his preferences on X as an individual preference ordering, $O^k = \{o^k(1), \ldots, o^k(n)\}$, where $o^k(\cdot)$ is a permutation function over the index set, $\{1, \ldots, n\}$, for the expert, e_k [4,21]. Therefore, according to this point of view , an ordered vector of alternatives, from best to worst, is given.

2. *A multiplicative preference relation.* In this case, the expert's preferences on X are described by a positive preference relation, $A^k \subset X \times X$. The intensity of preference, a^k_{ij}, is measured using a ratio scale, and in particular, as Saaty [19] showed, the 1–9 scale, $a^k_{ij} = 1$ indicates indifference between x_i and x_j, $a^k_{ij} = 9$ indicates that x_i is absolutely preferred to x_j, and $a^k_{ij} \in 2, 3, \ldots, 8$ indicates intermediate evaluations. The preference matrix, A^k, is assumed multiplicative reciprocal [18,19], i.e., $a^k_{ij} \cdot a^k_{ji} = 1$.
3. *A utility function.* In this case, an expert, e_k, gives his preferences on X as a set of n utility values, $U^k = \{u^k_i,\ i = 1, \ldots, n\}$, $u^k_i \in [0, 1]$, where u^k_i represents the utility evaluation given by the expert e_k to the alternative x_i [15,24].

In this context, the resolution process of the MPDM problem consists in obtaining a set of solution alternatives, $X_{\text{sol}} \subset X$, from the preferences given by the experts. As we assume that the experts give their preferences in different ways, the first step must be to obtain a uniform representation of the preferences.

In the following section, we will study how to make information uniform and thus obtain transformation functions.

3. Making the information uniform using multiplicative preference relations

As we said before, when the information provided by a group of experts is assumed to be of a diverse nature, we need to make the information uniform. As we pointed out at the beginning, preference relation is the most common representation of information used in MPDM problems because it is a useful tool in modelling decision processes, above all when we want to aggregate experts' preferences into group preferences [3,5,6,9,10,16,18,19,23,24]. Therefore, in this case, we propose to use the multiplicative preference relation as the base to uniform the information. Having said that, we need to find functions to transform preference ordering and utility values into multiplicative preference relations. This is dealt with in more detail in the subsections follow.

3.1. Utility values and multiplicative preference relation

Here we assume that an expert, e^k, gives his preferences on X by means of a set of utility values, given on the basis of a positive ratio scale, $U^k = \{u^k_i, i = 1, \ldots, n\}$, i.e., each alternative, x_i, is given a real number, $u^k_i \in [0, 1]$, indicating the performance of that alternative according to his point of view. For every set of utility values, U^k, we suppose, without loss of generality, that the higher the value, the better the alternative satisfies the expert.

Any possible transformation function, h, used to derive a multiplicative preference relation from a set of utility values, must obtain the preference value of the alternative x_i over x_j, a^k_{ij}, based only on the values u^k_i and u^k_j. Therefore, there exists a function from $[0, 1]^2$ to $[0, 1]$ such that

$$a^k_{ij} = h(u^k_i, u^k_j).$$

This transformation function, h, must verify that the higher u^k_i the higher a^k_{ij} and the higher u^k_j the lower a^k_{ij}, in other words function h must be increasing in the first argument and decreasing in the second argument. On the other hand, if the pair of values (u^k_i, u^k_j) change slightly, the preference between that pair of alternatives should change slightly too, i.e., the function h must be a continuous function.

An example of this type of transformation function is one that obtains the value of preferences based on the quotient between the respective utility values of the alternatives, i.e.,

$$h(u^k_i, u^k_j) = l\left(\frac{u^k_i}{u^k_j}\right),$$

where l is an increasing function. This type of function has been investigated by Luce and Suppes [15] and by Chiclana et al. [5] when studying the relationship between utility values and fuzzy preference relations.

Interpreting u^k_i/u^k_j as a ratio of the preference intensity for x_i to that of x_j, i.e., x_i is u^k_i/u^k_j times as good as x_j, and assuming a multiplicative preference relation, the simplest function to obtain the intensity of preference is the one proposed and used by Saaty in his AHP method [18,19], i.e.,

$$a_{ij}^k = h(u_i^k, u_j^k) = l^1\left(\frac{u_i^k}{u_j^k}\right) = \frac{u_i^k}{u_j^k},$$

being this function, as we will show, a particular case of a general family of transformation functions.

Without loss of generality, we suppose that the utility values belong to $[0, 1]$. Then, the transformation function, that we look for,

$$h : [0,1] \times [0,1] \to R^+$$

has to fulfil the following properties:
1. $h(z,y) \cdot h(y,z) = 1\ \forall z, y \in [0,1]$.
2. $h(z,z) = 1\ \forall z \in [0,1]$.
3. $h(z,y) > 1$ if $z > y\ \forall z, y \in [0,1]$.

- Property 1 is the multiplicative reciprocity condition.
- Property 2 is a result of property 1. It indicates the indifference that an expert shows between two alternatives, satisfying his criterion with the same intensity.
- Finally, property 3 indicates that between two alternatives, the expert gives a definite preference to the alternative with a higher utility value than the other. This is a result of property 2 and the fact that function, h, has to be increasing in the first argument and decreasing in the second argument.

Property 1 can be represented in the following equivalent parametric form:

$$h(z,y) = \tan^2 \gamma, \quad h(y,z) = \cot^2 \gamma,$$

where $\gamma \in [0, \pi/2]$ represents the value of the vector angle of a point with cartesian coordinates (z, y) or, in general, $(q(z), q(y))$, being

$$q : [0,1] \to R^+$$

is any increasing and continuous function. We then have

$$\gamma = \arctan \frac{q(z)}{q(x)}$$

and therefore

$$h(z,y) = \left[\frac{q(z)}{q(y)}\right]^2.$$

Finally, writing $s(z) = [q(z)]^2$, the expression of h becomes

$$h(z,y) = \frac{s(z)}{s(y)},$$

s being a continuous and increasing function.

We note that the above general form of transformation function of utility values is the one we would have obtained if we had assumed h is a separable function. This assumption is based on the fact that the set of utility values are given on the basis of a positive ratio scale. In fact, if

$$h(z,y) = s(z) \cdot r(y) \quad \forall z, y \in [0,1],$$

where s and r are functions with the same domain $[0, 1]$, same sign, increasing the first and decreasing the second, then from property 2 we have

$$s(y) \cdot r(y) = 1 \quad \forall z, y \in [0,1]$$

and therefore, $r(y) = 1/s(y) \quad \forall y \in [0,1]$, which is the same expression of function h we obtained above.

Summarising, we have the following results:

Proposition 1. *For every set of utility values, $U^k = \{u_1^k, \ldots, u_n^k\}$, over a set of alternatives, X, given on the basis of a positive ratio scale, the preference of alternative x_i over x_j, a_{ij}^k, is obtained by the following function h:*

$$a_{ij}^k = h(u_i^k, u_j^k) = \frac{s(u_i^k)}{s(u_j^k)},$$

where $s : [0,1] \to R^+$ is any continuous and increasing function.

Corollary 1. *When $s(u_i^k) = u_i^k$ the transformation function h reduces to the transformation function l^1 proposed by Saaty* [18,19].

As it was aforementioned, a particular case of transformation function is

$$h(u_i^k, u_j^k) = l\left(\frac{u_i^k}{u_j^k}\right),$$

where l is a continuous and increasing function. The following result gives the general expression of this type of function, with the one proposed by Saaty [18,19] as a particular case.

Proposition 2. *The common general solution of the following two functional equations*:
1. $h(z,y) \cdot h(y,z) = 1$,
2. $l(z/y) \cdot l(y/z) = 1$,
being h a function increasing in the first argument, decreasing in the second argument and continuous and l an increasing and continuous function, is

$$h(z,y) = l\left(\frac{z}{y}\right) = \left(\frac{z}{y}\right)^c,$$

where $c \in R^+$.

Proof. From the above proposition, we know that

$$h(z,y) = \frac{s(z)}{s(y)},$$

being s a continuous and increasing function, is the general solution of the first functional equation. As we try to find the common general solution to both functional equations, then

$$l\left(\frac{z}{y}\right) = \frac{s(z)}{s(y)}$$

implies

$$l(z) = l\left(\frac{z}{1}\right) = \frac{s(z)}{s(1)} \Rightarrow s(z) = s(1) \cdot l(z)$$
$$\Rightarrow l\left(\frac{1}{y}\right) = \frac{s(1)}{s(y)} = \frac{s(1)}{s(1) \cdot l(y)} = \frac{1}{l(y)}$$
$$\Rightarrow l(z \cdot y) = l\left(\frac{z}{\frac{1}{y}}\right) = \frac{s(z)}{s(\frac{1}{y})} = \frac{s(1) \cdot s(z)}{s(1) \cdot s(\frac{1}{y})} = \frac{l(z)}{l(\frac{1}{y})}$$
$$= l(z) \cdot l(y)$$

that is, $l(z \cdot y) = l(z) \cdot l(y) \quad \forall z, y \in [0,1]$. This means that the function l has to be

$$l(z) = z^c, \quad c > 0.$$

Therefore, the common general solution of the two above functional equations is

$$h\left(\frac{z}{y}\right) = \left(\frac{z}{y}\right)^c, \quad c > 0.$$

We note that if preferences between alternatives do not change when the utility values of the implied alternatives change in the same proportion, then the function obtained in the previous proposition is the one we have to use. In fact, if we suppose that

$$a_{ij}^k = h^2(u_i^k, u_j^k) = h^2(c \cdot u_i^k, c \cdot u_j^k)$$

then, taking $c = 1/u_j^k$ we have

$$h^2(u_i^k, u_j^k) = h^2\left(\frac{u_i^k}{u_j^k}, 1\right) = l\left(\frac{u_i^k}{u_j^k}\right)$$

and therefore, applying the previous result, it has to be

$$a_{ij}^k = \left(\frac{u_i^k}{u_j^k}\right)^c, \quad c > 0.$$

The relationship between utility values given on the basis of a difference scale and multiplicative preference relations is studied in the following subsection.

3.2. Preference ordering and multiplicative preference relation

In this case, we assume that an expert, e_k, gives his preferences on X by means of a preference ordering, $O^k = \{o^k(1), \ldots, o^k(n)\}$. For every preference ordering, O^k, we suppose, without loss of generality, that the lower the position of an alternative the better it satisfies the expert, and vice versa. For example, suppose that an expert, e_k, gives his preferences about a set of four alternatives, $X = \{x_1, x_2, x_3, x_4\}$, by means of the following preference ordering, $O^k = \{3, 1, 4, 2\}$, then alternative x_2 is the best one for that particular expert, while alternative x_3 is the worst.

Clearly, an alternative satisfies an expert e_k more or less depending on its position in his preference ordering O^k. In a similar way, we consider that for an expert, e_k, his preference value of the alternative x_i over x_j, a_{ij}^k, depends only on the values of $o^k(i)$ and $o^k(j)$, i.e., we assert that there exists a transformation function, f, that assigns an intensity value of preference of any alternative over any other alternative, from any preference ordering,

$$a_{ij}^k = f(o^k(i), o^k(j)).$$

This transformation function, f, must verify that the higher $o^k(i)$ the lower a_{ij}^k and the higher $o^k(j)$ the higher a_{ij}^k. In other words, function f must be decreasing in the first argument and increasing in the second argument.

An example of this type of transformation function is one that obtains the intensity value of preference based on the value of the difference between the alternatives' positions, i.e.,

$$f(o^k(i), o^k(j)) = g(o^k(j) - o^k(i)),$$

being g is an increasing function. The easiest and simplest example of this type of function is the following:

$$a_{ij}^k = g^1(o^k(j) - o^k(i)) = \begin{cases} 9 & \text{if } o^k(j) > o^k(i), \\ 1/9 & \text{if } o^k(i) > o^k(j), \end{cases} \quad i \neq j.$$

In our example this transformation function gives the following preference relation:

$$A^k = \begin{bmatrix} - & 1/9 & 9 & 1/9 \\ 9 & - & 9 & 9 \\ 1/9 & 1/9 & - & 1/9 \\ 9 & 9 & 9 & - \end{bmatrix}.$$

Its simplicity and easy use are the only virtues of this particular transformation function, g^1. However, this preference relation does not reflect the case when an expert is not able to distinguish between two alternatives, i.e., when there is an indifference between two alternatives. This can be easily solved with an extension of this function, g^1, as follows:

$$a_{ij}^k = g^2(o^k(j) - o^k(i)) = \begin{cases} 9 & \text{if } o^k(j) - o^k(i) > 0, \\ 1 & \text{if } o^k(j) - o^k(i) = 0, \\ 1/9 & \text{if } o^k(j) - o^k(i) < 0, \end{cases} \quad i \neq j.$$

In any case, both functions, g^1 and g^2, do not reflect any kind of intensity of preference between alternatives. In our example, they do not distinguish between the preference of alternative x_2 over x_4 and the preference of alternative x_2 over x_3, although this should not be the case. To deal with these situations we need to use another type of function which reflects appropriately the different positions between alternatives. For example, if $a_{24}^k = a$ then a_{41}^k and a_{13}^k should be equal to a, but a_{21}^k should be greater than or equal to a and less than or equal to a_{23}^k.

This new type of function can be obtained, for example, by giving a value of importance or utility to each alternative, in such a way that the lower the position of an alternative, the higher the value of utility. We can assume that the preference of the best alternative over the worst alternative is the maximum allowed, that is 9. So if, for example, $o^k(i) = 1$ and $o^k(j) = n$, then we assume that $a_{ij}^k = 9$. As we said before, the utility value, u_i^k, associated to alternative, x_i, depends on the value of its position $o^k(i)$, in such a way that the bigger the value of $n - o^k(i)$, the bigger the value of u_i^k, i.e.

$$u_i^k = v(n - o^k(i)),$$

where v is a non-decreasing function. An example of this function is

$$u_i^k = v(n - o^k(i)) = \frac{n - o^k(i)}{n - 1}.$$

It is clear that the maximum utility value corresponds to the first alternative and the minimum utility value to the last alternative in the preference ordering. In this context, we have a normalised set of n utility values, that is,

$$\mathrm{MAX}_i\{u_i^k\} - \mathrm{MIN}_i\{u_i^k\} \leqslant 1.$$

Therefore, we have utility values given on the basis of a difference scale.

We are looking for a general expression of the transformation function of a preference ordering into a multiplicative preference relations, f, in such a way, that given a pair of alternatives, (x_i, x_j), of which we only know their position numbers in that preference ordering, $(o^k(i), o^k(j))$, it gives us the preference of x_i over x_j, a^k_{ij}. This transformation function, f, as was aforementioned, must be decreasing in the first argument and increasing in the second argument and has to fulfil the following properties:

1. $f(o^k(i), o^k(j)) \in [1/9, 9] \; \forall i, j.$
2. $f(o^k(i), o^k(j)) \cdot f(o^k(j), o^k(i)) = 1 \; \forall i, j.$
3. $f(o^k(i), o^k(j)) = 1$ if $o^k(i) = o^k(j) \; \forall i, j.$
4. $f(o^k(i), o^k(j)) > 1$ if $o^k(i) < o^k(j) \; \forall i, j.$

In this case, as the values are given on the basis of a difference scale, there exists a function, g, such that

$$a^k_{ij} = f(o^k(i), o^k(j)) = g(o^k(j) - o^k(i)),$$

where g is an increasing function. In what follows, both $o^k(i)$ and $o^k(j)$ can be replaced by $u^k_i = v(n - o^k(i))$ and $u^k_j = v(n - o^k(j))$, respectively, being v an increasing function, and the difference $o^k(j) - o^k(i)$ by the difference $u^k_i - u^k_j$. Furthermore, g must fulfil

1. $g(z) \in [1/9, 9]$.
2. $g(z) \cdot g(-z) = 1$
3. $g(0) = 1$.
4. $g(z) > 1$ if $z > 0$.

Property 2 can be expressed as follows:

$$g(z) \cdot g(-z) = g(0),$$

which implies

1. $g(z) \cdot g(y) \cdot g(-z) \cdot g(-y)$
   $= [g(z) \cdot g(-z)] \cdot [g(y) \cdot g(-y)]$
   $= 1 \cdot 1 = 1 = g(0)$, and
2. $g(z + y) \cdot g(-z - y) = g(0)$.

That is

$$g(z + y) \cdot g(-z - y) = g(z) \cdot g(y) \cdot g(-z) \cdot g(-y)$$

and by symmetry, only one of the following four functional equations is true:

1. $g(z + y) = g(z) \cdot g(y)$.
2. $g(z + y) = g(-z) \cdot g(y)$.
3. $g(z + y) = g(z) \cdot g(-y)$.
4. $g(z + y) = g(-z) \cdot g(-y)$.

We are going to demonstrate that the only possible functional equation is the first one. In fact, the other three functional equations have only one possible solution $g(z) = 1 \; \forall \; z$, which is nonsense because this function does not verify property 4. If we suppose that $g(z + y) = g(-z) \cdot g(y)$. Taking $y = 0$ we have $g(z) = g(-z)$, and applying the reciprocity property it has to be $g(z) = 1 \; \forall \; z$. Therefore, the option second functional equations is not possible. In a similar way, we can prove that the other two are not possible either. So, function g verifies the first condition, i.e.,

$$g(z + y) = g(z) \cdot g(y) \quad \forall z.$$

It is well-known that the general solution of this functional equation is $g(z) = \exp_a z$ [1]. From $g(1) = 9$, it results that so that the function we are looking for is

$$g(z) = 9^z \quad \forall z.$$

Summarising, we have the following result:

Proposition 3. *Suppose we have a set of alternatives, X, and associated with it a preference ordering, $O^k = \{o^k(1), \ldots, o^k(n)\}$. Then, the preference degree of alternative x_i over x_j, a^k_{ij}, is given by the following transformation function, f^1:*

$$a^k_{ij} = f^1(o^k(i), o^k(j)) = 9^{u^k_i - u^k_j},$$

where $u^k_i = v(n - o^k(i))$ and $u^k_j = v(n - o^k(j))$ are utility values associated to alternatives x_i and x_j, respectively, by means of an increasing function v.

4. The MPDM based on fuzzy majority for different preference structures

In this section, we present a multiplicative decision model assuming that we have different preference structures to represent the experts' opinions, i.e., preference orderings, utility functions and multiplicative preference relations. The two new features of this multiplicative decision model are:

- It integrates different preference structures via the transformation functions obtained in Section 3.

- It is based on the concept of fuzzy majority and thus, all decisions are achieved according to a majority of people's opinions.

The multiplicative decision model is developed following two steps (see Fig. 1):

1. *Making the information uniform*. For every preference ordering and set of utility values we derive a multiplicative preference relation. To do this, we use the transformation functions defined in Section 3.

2. *Application of a selection process*. Once we have the information given represented uniformly we apply a selection process to choose the best alternatives. This selection process is developed in two phases:

(a) *Aggregation phase*. A collective multiplicative preference relation is obtained from all the individual multiplicative preference relations, using an OWG operator, which implements the concept of fuzzy majority.

(b) *Exploitation phase*. Using again the concept of fuzzy majority, but in another sense, two choice degrees of alternatives are applied, the quantifier guided dominance degree and the quantifier guided non-dominance degree. These choice degrees act over the collective preference relation supplying a selection set of alternatives.

In the following subsections, we analyse each phase of the selection process.

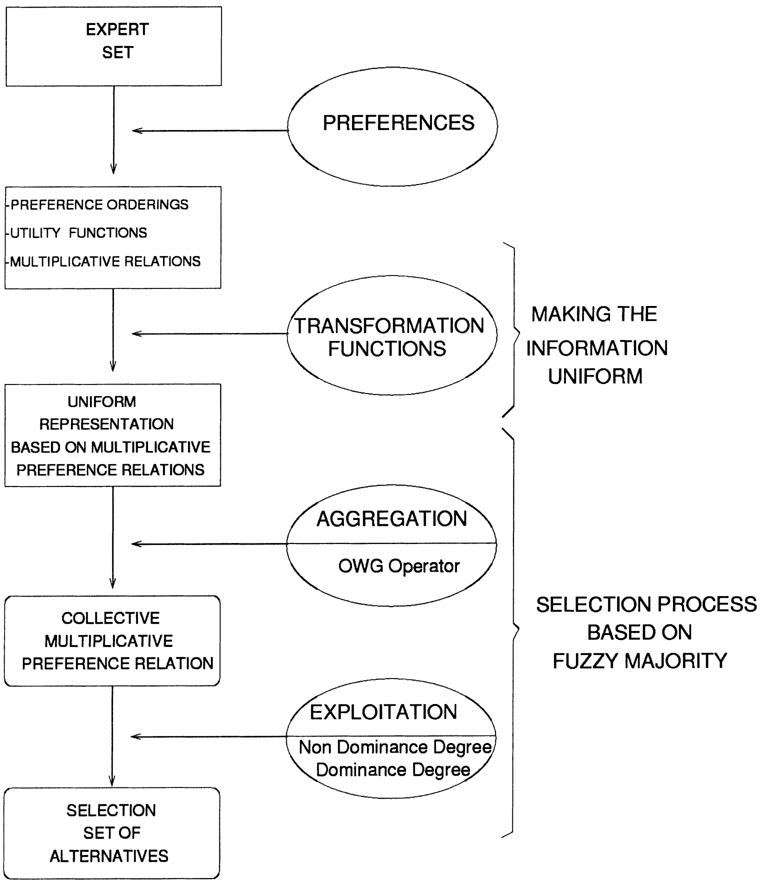

Fig. 1. Resolution process of the MPDM problem.

4.1. Aggregation: The collective multiplicative preference relation

Once we have made the information uniform, we have a set of m individual multiplicative preference relations, $\{A^1, \ldots, A^m\}$. From this set of relations we derive the collective multiplicative preference relation, A^c. Each value, $a_{ij}^c \in [1/9, 9]$, represents the preference of alternative x_i over alternative x_j according to the majority of the experts' opinions. Traditionally, the majority is defined as a threshold number of individuals. Fuzzy majority is a soft majority concept expressed by a fuzzy linguistic quantifier [28] (see Appendix). To calculate each a_{ij}^c we will use the OWG operator, as the aggregation operator of information.

Definition 1. Let $\{a_1, a_2, \ldots, a_m\}$ be a list of values to aggregate, then, an OWG operator of dimension m is a function ϕ^G,

$$\phi^G : R^m \rightarrow R$$

which has associated a set of weights W and is defined as

$$\phi^G(a_1, a_2, \ldots, a_m) = \prod_{k=1}^{m} c_k^{w_k},$$

where $W = [w_1, \ldots, w_m]$, is an exponential weighting vector, such that, $w_i \in [0, 1]$ and $\sum_k w_k = 1$ and C is the associated ordered value vector. Each element $c_i \in C$ is the ith largest value in the collection $\{a_1, \ldots, a_m\}$.

The OWG operator reflects the fuzzy majority calculating its weighting vector by means of a fuzzy linguistic quantifier according to Yager's ideas [27]. In the case of a non-decreasing proportional quantifier Q, the weighting vector is calculated using the following expression:

$$w_k = Q(k/m) - Q((k-1)/m), \quad k = 1, \ldots, m.$$

When a fuzzy quantifier Q is used to calculate the weights of the OWG operator ϕ^G, it is represented by ϕ_Q^G.

Therefore, the collective multiplicative preference relation is obtained as follows:

$$a_{ij}^c = \phi_Q^G(a_{ij}^1, \ldots, a_{ij}^m).$$

4.2. Exploitation: Choosing the alternative(s)

The exploitation phase consists of choosing the alternative(s) "best" acceptable to the group of individuals as a whole. To do so, we use two quantifier guided choice degrees of alternatives, defined over the collective multiplicative preference relation and based on the concept of fuzzy majority: *a quantifier guided dominance degree and a quantifier guided non dominance degree.* Both choice degrees are calculated from A^c using the OWG operator as follows:

1. *Quantifier guided dominance degree*. For the alternative, x_i, we calculate the quantifier guided dominance degree, MQGDD_i, from A^k as follows:

$$\text{MQGDD}_i = \frac{1}{2} \cdot (1 + \log_9 \phi_Q^G(a_{ij}^c,\ j = 1, \ldots, n)).$$

   It is used to quantify the dominance that one alternative has over all the others in a fuzzy majority sense.
2. *Quantifier guided non-dominance degree*. We also compute the quantifier guided non dominance degree, MQGNDD_i, according to the following expression:

$$\text{MQGNDD}_i^k = 1 + \log_9 \phi_Q^G(r_{ij}^c,\ j = 1, \ldots, n),$$

   where r_{ij}^c is a preference value obtained as $r_{ij}^c = \min\{a_{ij}^c, 1\}$. In our context, MQGNDD_i gives the degree in which each alternative is not dominated by a fuzzy majority of the set of alternatives.

In order to obtain the selection set of alternatives we may apply these choice degrees according to the following selection scheme [5]:

- *Step 1*. The application of each choice degree of alternatives over X allows us to obtain the following sets of alternatives:

$$X^{\text{MQGDD}} = \left\{ x_i \mid x_i \in X, \text{MQGDD}_i = \sup_{x_j \in X} \text{MQGDD}_j \right\}$$

$$X^{\text{MQGNDD}} = \left\{ x_i \mid x_i \in X, \text{MQGNDD}_i = \sup_{x_j \in X} \text{MQGNDD}_j \right\}$$

whose elements are called maximum dominance elements and maximal non-dominated elements, respectively.

- *Step 2*. The application of the conjunction selection policy obtains the following set of alternatives:

  $$X^{\text{QGCP}} = X^{\text{MQGDD}} \cap X^{\text{MQGNDD}}.$$

  If $X^{\text{QGCP}} \neq \emptyset$, then End and this is the solution. Otherwise continue,
- *Step 3*. The application of one sequential selection policy, according to either a dominance or non dominance criterion, i.e.,
  - *Dominance based sequential selection process* MQG-DD-NDD.
    To apply the quantifier guided dominance degree over *X*, and obtain X^{MQGDD}. If $\#(X^{\text{MQGDD}}) = 1$ then End, and this is the solution set. Otherwise, continue obtaining

    $$X^{\text{MQG-DD-NDD}} = \left\{ x_i \mid x_i \in X^{\text{MQGDD}}, \text{MQGNDD}_i = \sup_{x_j \in X^{\text{MQGDD}}} \text{MQGNDD}_j \right\}.$$

    This is the selection set of alternatives.
  - *Non-dominance based sequential selection process MQG-NDD-DD*.
    To apply the quantifier guided non dominance degree over *X*, and obtain X^{MQGNDD}. If $\#(X^{\text{MQGNDD}}) = 1$ then End, and this is the solution set. Otherwise, continue obtaining

    $$X^{\text{MQG-NDD-DD}} = \left\{ x_i \mid x_i \in X^{\text{MQGNDD}}, \text{MQGDD}_i = \sup_{x_j \in X^{\text{MQGNDD}}} \text{MQGDD}_j \right\}.$$

    This is the selection set of alternatives.

4.3. Consistency of decision model

In this subsection, we analyse the consistency of the multiplicative decision model built using the transformation functions given in Propositions 1–3 and both quantifier guided choice degrees. In particular, we show that the transformation functions act coherently according to both choice degrees, because the ranking among the alternatives that we can obtain from any of the considered representation structures (preference ordering and utility function) is not affected if we apply any of the two choice degrees on the respective multiplicative preference relations obtained via the transformation functions.

The following proposition expresses this condition.

Proposition 4. *Let* $U^k = \{u_1^k, \ldots, u_n^k\}$ $(O^k = \{o^k(1), \ldots, o^k(n)\})$ *be a set of utility values (preference orderings) assigned by an expert* e_k *to a set of alternatives X. Let* A^k *be the multiplicative preference relation obtained according to the transformation function given in Proposition* 1 *or* 2 *(Proposition* 3*). If* $u_j^k \geqslant u_i^k$ $(o^k(i) \geqslant o^k(j))$ *then the choice degrees* $(\text{MQGDD}_i^k, \text{MQGNDD}_i^k, \text{MQGDD}_j^k, \text{MQGNDD}_j^k)$ *obtained from* A^k *satisfy the following relationships*:
1. $\text{MQGDD}_i^k \leqslant \text{MQGDD}_j^k$ and
2. $\text{MQGNDD}_i^k \leqslant \text{MQGNDD}_j^k$.

Proof. The demonstration is a result of the OWG operator and the transformation functions being non-decreasing functions. □

5. Example

We present an example to illustrate the decision model studied in this paper. Suppose that we have a set of six experts, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, and a set of four alternatives, $X = \{x_1, x_2, x_3, x_4\}$. Suppose that experts e_1, e_2 give their opinions in terms of preference ordering, experts e_3, e_4 in terms of utility values, and experts e_5, e_6 in terms of multiplicative preference relations. The information is the following:

- e_1: $O^1 = \{3, 1, 4, 2\}$.
- e_2: $O^2 = \{3, 2, 1, 4\}$.
- e_3: $U^3 = \{0.5, 0.7, 1, 0.1\}$.
- e_4: $U^4 = \{0.7, 0.9, 0.6, 0.3\}$.
- e_5:

$$A^5 = \begin{bmatrix} 1 & 1/5 & 1/5 & 3 \\ 5 & 1 & 5 & 7 \\ 5 & 1/5 & 1 & 5 \\ 1/3 & 1/7 & 1/5 & 1 \end{bmatrix}.$$

- e_6:

$$A^6 = \begin{bmatrix} 1 & 1 & 5/2 & 9 \\ 1 & 1 & 4 & 3/2 \\ 2/5 & 1/4 & 1 & 4 \\ 1/9 & 2/3 & 1/4 & 1 \end{bmatrix}.$$

5.1. Making the information uniform

Using transformation functions f^1 with $u_i^k = v(n - o^k(i)) = (n - o^k(i))/(n-1)$ and l^1 to make the information uniform, we have the following multiplicative preference relations:

$$A^1 = \begin{bmatrix} 1 & 9^{-2/3} & 9^{1/3} & 9^{-1/3} \\ 9^{2/3} & 1 & 9 & 9^{1/3} \\ 9^{-1/3} & 1/9 & 1 & 9^{2/3} \\ 9^{1/3} & 9^{-1/3} & 9^{-2/3} & 1 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} 1 & 9^{-1/3} & 9^{-2/3} & 9^{1/3} \\ 9^{1/3} & 1 & 9^{-1/3} & 9^{2/3} \\ 9^{2/3} & 9^{1/3} & 1 & 9 \\ 9^{-1/3} & 9^{-2/3} & 1/9 & 1 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} 1 & 5/7 & 1/2 & 5 \\ 7/5 & 1 & 1/7 & 7 \\ 2 & 7 & 1 & 10 \\ 1/5 & 1/7 & 1/10 & 1 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 1 & 7/9 & 7/6 & 7/3 \\ 9/7 & 1 & 3/2 & 3 \\ 6/7 & 2/3 & 1 & 2 \\ 3/7 & 1/3 & 1/2 & 1 \end{bmatrix}.$$

5.2. Selection process

5.2.1. Aggregation

Using the fuzzy majority criterion with the fuzzy linguistic quantifier *"at least half"*, with the pair $(0, 0.5)$, and the corresponding OWG operator with the weighting vector, $W = [1/3, 1/3, 1/3, 0, 0, 0]$, the collective multiplicative preference relation is:

$$A^c = \begin{bmatrix} 1 & 0.822 & 1.824 & 5.130 \\ 3.557 & 1 & 5.646 & 5.278 \\ 3.511 & 2.133 & 1 & 7.663 \\ 1.326 & 0.420 & 0.307 & 1 \end{bmatrix}.$$

5.2.2. Exploitation

We apply the exploitation process with the fuzzy quantifier "*most*" with the pair $(0.3, 0.8)$, i.e., the corresponding OWG operator with the weighting vector $W = [0, 0.4, 0.5, 0.1]$. The quantifier guided choice degrees of alternatives acting over the collective multiplicative preference relation supply the following values:

	x_1	x_2	x_3	x_4
MQGDD_i	0.5504	0.7964	0.7005	0.576
MQGNDD_i	0.9911	1	1	0.7489.

These values represent the dominance that one alternative has over "most" alternatives according to "at least half" of the experts, and the non-dominance degree to which the alternative is not dominated by "most" alternatives according to "at least half" of the experts, respectively. Clearly the maximal sets are:

$$X^{\text{MQGDD}} = \{x_2\} \quad \text{and} \quad X^{\text{MQGNDD}} = \{x_2, x_3\},$$

therefore, the selection set of alternatives according to the selection procedure is the singleton $\{x_2\}$.

6. Conclusions

In this paper, we have studied an MPDM problem assuming that the information is provided by means of different preference structures: preference orderings, utility values and multiplicative preference relations. We have presented a decision model to choose the best alternatives using multiplicative preference relation as the element base for that uniformation. The two main features of the decision model presented are:

- It integrates different preference structures by means of transformation functions that relate the preference structures.
- It is guided by fuzzy majority.

As a consequence, this multiplicative MPDM model provides a flexible framework to manage different structures of preferences, constituting an approximate decision model to real decision situations with experts from different areas of knowledge.

Appendix A. Fuzzy majority and fuzzy linguistic quantifiers

The majority is traditionally defined as a threshold number of individuals. Fuzzy majority is a soft majority concept expressed by a fuzzy linguistic quantifier, which is manipulated via a fuzzy logic based calculus of linguistically quantified propositions.

Quantifiers can be used to represent the amount of items satisfying a given predicate. Classic logic is restricted to the use of only two quantifiers, *there exists* and *for all*, which are closely related to the *or* and *and* connectives, respectively. Human discourse is much richer and more diverse in its quantifiers, e.g., *about 5, almost all, a few, many, most, as many as possible, nearly half, at least half*. In an attempt to bridge the gap between formal systems and natural discourse and, in turn, to provide a more flexible knowledge representation tool, Zadeh introduced the concept of fuzzy quantifiers [28]. Zadeh suggested that the semantic of a fuzzy quantifier can be captured by using fuzzy subsets for its representation. He distinguished between two types of fuzzy quantifiers, *absolute* and *proportional or relative*. Absolute quantifiers are used to represent amounts that are absolute in nature such as *about 2* or *more than 5*. These absolute linguistic quantifiers are closely related to the concept of count or number of elements. He defined these quantifiers as fuzzy subsets of the non negative real numbers, $\mathscr{R}^+$. In this approach, an absolute quantifier can be represented by a fuzzy subset Q, such that for any $r \in \mathscr{R}^+$ the membership degree of r in Q, $Q(r)$, indicates the degree to which the amount r is compatible with the quantifier represented by Q. Proportional quantifiers, such as *most, at least half*, can be represented by fuzzy subsets of the

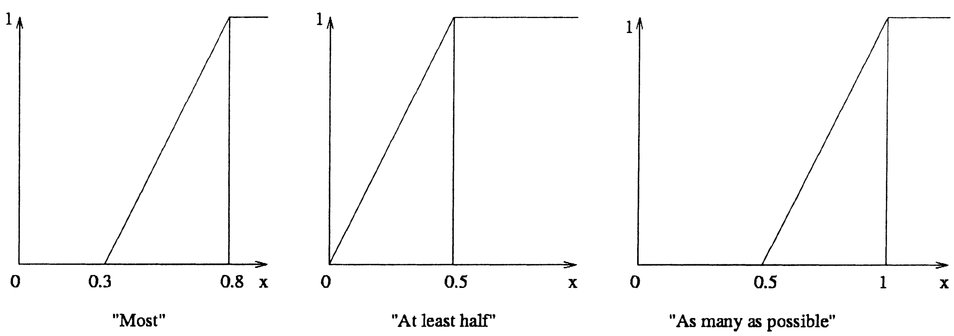

Fig. 2. Proportional fuzzy quantifiers.

unit interval, [0,1]. For any $r \in [0,1]$, $Q(r)$ indicates the degree to which the proportion r is compatible with the meaning of the quantifier it represents. Any quantifier of natural language can be represented as a proportional quantifier or given the cardinality of the elements considered, as an absolute quantifier. Functionally, fuzzy quantifiers are of three types, *increasing, decreasing,* and *unimodal*. An increasing quantifier is characterised by the relationship $Q(r_1) \geqslant Q(r_2)$ if $r_1 > r_2$. Examples of these quantifiers are *most, at least half*. A decreasing quantifier is characterised by the relationship $Q(r_1) \leqslant Q(r_2)$ if $r_1 < r_2$.

An absolute quantifier $Q : \mathscr{R}^+ \rightarrow [0,1]$ satisfies: $Q(0) = 0$, exists $k \mid Q(k) = 1$. A relative quantifier, $Q : [0,1] \rightarrow [0,1]$, satisfies: $Q(0) = 0$, exists $r \in [0,1] \mid Q(r) = 1$. A non decreasing quantifier satisfies: $\forall\ a,b$ if $a > b$ then $Q(a) \geqslant Q(b)$.

The membership function of a non-decreasing relative quantifier can be represented as

$$Q(r) = \begin{cases} 0 & \text{if } r < a, \\ \frac{r-a}{b-a} & \text{if } a \leqslant r \leqslant b, \\ 1 & \text{if } r > b \end{cases}$$

with $a, b, r \in [0,1]$. Some examples of proportional quantifiers are shown in Fig. 2, where the parameters, (a,b) are $(0.3, 0.8)$, $(0, 0.5)$ and $(0.5, 1)$, respectively.

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