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Theory and Methodology

Choice functions and mechanisms for linguistic preference relations

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Abstract

The problem of finding a solution set of alternatives when a linguistic preference relation represents a collective preference is analyzed following two research lines, choice functions and mechanisms.

Four classical choice sets of alternatives for a linguistic preference relation are presented and some relations between them are pointed out. The concept of linguistic choice function as a tool to derive linguistic choice sets of alternatives is introduced and some particular linguistic choice functions are presented together with a study of their rationality properties. On the other hand, two types of linguistic choice mechanisms to derive solution sets of alternatives from linguistic choice functions are introduced: simple and composite ones.

The concept of linguistic covering relation is introduced with a view to allow us the design of consistent linguistic choice mechanisms which may achieve more precise and coherent solution sets of alternatives.

Keywords: Linguistic modeling; Linguistic preference relation; Choice functions

1. Introduction

Fuzzy preference relations seem to be a useful tool in modelling decision processes to compare and evaluate alternatives or candidates from ill-known or ill-defined preference information [6,16]. Among others, they have been widely used in group decision making processes [13,14]. Sometimes, experts are not able to estimate their performance judgements about alternatives with exact numerical values and they use linguistic values. Then, we consider that they provide their preferences by means of *linguistic preference relations* using a set of linguistic terms S [5,8,24].

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Usually, in a fuzzy environment a group decision problem is taken out as follows. It is assumed that there exists a finite set of alternatives $X = \{x_1, \dots, x_n\}$ as well as a finite set of experts $E = \{e_1, \dots, e_m\}$. Each expert $e_k \in E$ provides his performance judgements on X by means of a fuzzy preference relation $P^k \subset X \times X$, characterized by a membership function $\mu_{P^k}(x_i, x_j) \in [0, 1]$, denoting the degree of preference of alternative x_i over x_j. Then, the question consists of finding a solution set of alternatives from the set of fuzzy preference relations $\{P^1, \dots, P^m\}$. It may be solved in two phases [11,19]: an *aggregation phase* of the individual fuzzy preference relations $\{P^1, \dots, P^m\}$ into collective fuzzy preference relation P^c followed by an *exploitation phase* of P^c which obtains a choice set of alternatives or rank ordering among the alternatives, and then, derives the solution set of alternatives from the choice set.

In decision theory the exploitation phase is modeled using *choice functions* to separate the best alternatives from a crisp collective preference relation [2,20]. In a fuzzy context different authors have proposed various *fuzzy choice functions* to find the choice set of alternatives from a fuzzy collective preference relation. An overview of more important results on fuzzy choice functions can be found in Refs. [18,21].

In this paper, we present an analysis of the problem of finding the choice and solution sets of alternatives under the use of linguistic preference relations to represent ill-known or ill-defined preference information. We analyze both questions and for each one we obtain the following issues:

- *On the choice sets of alternatives*. Following Roubens's and Switalski's studies [18,21] we define for a linguistic preference relation four classical choice sets of alternatives and investigate connections between them: *a linguistic choice set of greatest alternatives, a linguistic choice set of nondominated alternatives, a linguistic choice set of strictly greatest alternatives* depending on some linguistic conjunction functions, and *a linguistic choice set of maximal alternatives* depending on some linguistic implication functions. We also introduce the concept of *linguistic choice function* associated with a linguistic preference relation as a generalization of linguistic choice sets, present various particular linguistic choice functions and analyze them showing their rationality properties.
- *On the solution sets of alternatives*. In general, we point out the existence of a problem of specificity and consensus in the individual application of choice functions, that is, in many cases we find that the solution sets of alternatives provided by different choice functions are very general and different. Therefore, we propose to solve it by means of the distinction between two types of linguistic choice mechanisms or ways of application of choice functions: *simple linguistic choice mechanisms*, which derives the solution set of alternatives by means of one choice function, and *composite ones*, which uses various choice functions. On the other hand, we also must point out that due to the lack of transitivity property of the collective linguistic preference relation, sometimes a problem of consistency appears in the solution set of alternatives. Following Perny's studies [17] we introduce to solve it the concept of *linguistic covering relation*, as a way to get a "transitive linguistic preference relation" from given one. In this way, we propose to develop *consistent linguistic choice mechanisms* to find more precise and coherent solution sets of alternatives.

To do so, the paper is structured as follows. Section 2 presents the linguistic preference relations to express preferences. Section 3 defines and analyzes four classical choice sets of alternatives for a linguistic preference relation. Section 4 introduces the concept of linguistic choice function, presents various linguistic choice functions and studies their rationality properties. Section 5 shows different linguistic choice mechanisms and characterizes consistent linguistic choice mechanisms and, finally, the last section contains our concluding remarks.

2. Linguistic preference relations to express preferences

The use of fuzzy preference relations in decision making problems to voice experts' opinions about a set of alternatives appears to be a useful tool to structure such a problem and to systematize the analysis of preferences. The *linguistic preference relations* are a kind of fuzzy preference relations used when experts are not able to estimate their preferences with exact numerical values and, then, they prefer to provide them by means of the linguistic values. In these cases, to fix previously a linguistic term set is absolutely essential to voice the experts' preferences.

We consider a finite and totally ordered linguistic term set $S = \{s_i\}, i \in H = \{0, \ldots, T\}$, with an odd cardinal and the mid term representing an assessment of "approximately 0.5", and with the rest of the terms being placed symmetrically around it as in Ref. [3]. We also assume that the limit of granularity is 11 or no more than 13. The semantic of linguistic terms is given by means of fuzzy numbers defined on the [0,1] interval, which are described by linear trapezoidal membership functions. This representation is achieved by the 4-tuple $(a_i, b_i, \alpha_i, \beta_i)$ (the first two paremeters indicate the interval in which the membership value is 1.0; the third and fourth parameters indicate the left and right widths of the distribution). Futhermore, we require the following properties:

1. The set is ordered: $s_i \geqslant s_j$ if $i \geqslant j$.
2. Negation operator: $\mathrm{Neg}(s_i) = s_j,\ j = T - i$.
3. Maximization operator: $\max(s_i, s_j) = s_i$ if $s_i \geqslant s_j$.
4. Minimization operator: $\min(s_i, s_j) = s_i$ if $s_i \leqslant s_j$.

For example, consider the following set of nine linguistic terms with its associated semantic [3]:

C	Certain	(1, 1, 0, 0)
EL	Extremely_likely	(0.98, 0.99, 0.05, 0.01)
ML	Most_likely	(0.78, 0.92, 0.06, 0.05)
MC	Meaningful_chance	(0.63, 0.80, 0.05, 0.06)
IM	It_may	(0.41, 0.58, 0.09, 0.07)
SC	Small_chance	(0.22, 0.36, 0.05, 0.06)
VLC	Very_low_chance	(0.1, 0.18, 0.06, 0.05)
EU	Extremely_unlikely	(0.01, 0.02, 0.01, 0.05)
I	Impossible	(0, 0, 0, 0)

Definition 1. Let $X = \{x_i,\ i = 1, \ldots, n\ (n \geqslant 2)\}$ be a finite set of alternatives, then a linguistic preference relation R is a fuzzy set in X^2 characterized by a membership function,

$$\mu_{\mathrm{R}} : X \times X \to S,$$

$$\mu_{\mathrm{R}}(x_i, x_j) = r_{ij}, \quad \forall x_i,\ x_j \in X,$$

indicating the linguistic preference degree of alternative x_i over x_j, i.e., $s_0 \leqslant r_{ij} \leqslant s_T$.

A linguistic preference relation R can be characterized by some of the following properties:

1. Reflexive: $r_{ii} = s_T,\ \forall i$.
2. Irreflexive: $r_{ii} = s_0,\ \forall i$.
3. Symmetric: $r_{ij} = r_{ji},\ \forall i, j$.
4. Antisymmetric: $\min(r_{ij}, r_{ji}) = s_0,\ \forall i, j, i \neq j$.
5. Complete: $\max(r_{ij}, r_{ji}) = s_T,\ \forall i, j, i \neq j$.
6. Transitive: $r_{ik} \geqslant \min(r_{ij}, r_{jk}),\ \forall i, j, k$.
7. Negatively transitive: $r_{ik} \leqslant \max(r_{ij}, r_{jk}),\ \forall i, j, k$.

3. Choice sets of alternatives for linguistic preference relations

In this section, we define various choice sets of alternatives for a linguistic preference relation according to Roubens's and Switalski's studies [18,21]. Firstly, we study some classical choice sets for binary preference relations, and later, we present their linguistic versions.

3.1. Classical choice sets of alternatives

Definition 2. Let X a finite set of alternatives and $P \in X^2$ be a (crisp) binary relation in X, then the upper x_i-cut and the lower x_i-cut of P are two crip sets defined in Ref. [21], respectively, by

$$P^+(x_i) = \{x_j \in X \mid x_j P x_i\} \quad \text{and} \quad P^-(x_i) = \{x_j \in X \mid x_i P x_j\}.$$

Therefore, we can obtain from P in terms of the P^+ and P^- the following four classical choice sets of alternatives.

Definition 3. The set of greatest alternatives for P in X is given by

$$G(X,P) = X \bigcap \left(\bigcap_{x_i \in X} P^+(x_i) \right).$$

Definition 4. The set of nondominated alternatives for P in X is given by

$$N(X,P) = X \bigcap \left(\bigcap_{x_i \in X} \overline{P^-(x_i)} \right),$$

where $\overline{P^-(x_i)}$ denotes the usual complement in X.

Definition 5. The set of maximal alternatives for P in X is given by

$$M(X,P) = X \bigcap \left(\bigcap_{x_i \in X} \left(P^+(x_i) \bigcup \overline{P^-(x_i)} \right) \right).$$

Definition 6. The set of strictly greatest alternatives for P in X is given by

$$S(X,P) = X \bigcap \left(\bigcap_{x_i \in X} \left(P^+(x_i) \bigcap \overline{P^-(x_i)} \right) \right).$$

As it is pointed out in Ref. [21], we may find the following relations.

Proposition 1. *For any binary relation P we have*:
1. $N(X,P) = G(X, \overline{P^{-1}})$,
2. $M(X,P) = G(X, P \bigcup \overline{P^{-1}})$
3. $S(X,P) = G(X, P \bigcap \overline{P^{-1}})$,

where $P^{-1} = \{(x_i, x_j) \mid x_j P x_i\}$ denotes the inverse relation.

3.2. Linguistic choice sets of alternatives

In Refs. [18,21] a definition of fuzzy choice sets for a conventional (numerical) fuzzy preference relation is presented which generalizes the classical definitions. Here, we extend those definitions to characterize the choice sets for a linguistic preference relation. We present four generalized linguistic choice sets of alternatives.

Definition 7. The upper x_i-cut and the lower x_i-cut of a linguistic preference relation R are two fuzzy sets in X defined, respectively, by

$$R^+(x_i) = \{(x_j, r_{ji}),\ \forall x_j \in X\} \quad \text{and} \quad R^-(x_i) = \{(x_j, r_{ij}),\ \forall x_j \in X\}.$$

Definition 8. The linguistic choice set of greatest alternatives for R in X is a fuzzy set in X given by

$$G^{\mathrm{L}}(X,R) = \nabla_{x_i \in X} R^+(x_i) = \{(x_j, \mu_{G^{\mathrm{L}}(X,R)}(x_j)),\ j = 1,\ldots,n\},$$

where $\nabla : S^n \to S$ denotes a combination operator of linguistic information. $\mu_{G^{\mathrm{L}}(X,R)}$ is the characteristic membership function of $G^{\mathrm{L}}(X,R)$ that assigns a linguistic degree of "greatestness" to each alternative $x_j \in X$ with respect to R according to the next expression:

$$\mu_{G^{\mathrm{L}}(X,R)} : X \to S, \quad \mu_{G^{\mathrm{L}}(X,R)}(x_j) = \nabla(r_{ji},\ i = 1,\ldots,n).$$

Some examples of operators ∇ can be found in Refs. [7,12,22,23]. Classically, it is used an operator modeling the conjunctions [18,21].

Definition 9. The linguistic choice set of nondominated alternatives for R in X is a fuzzy set in X given by

$$N^{\mathrm{L}}(X,R) = \nabla_{x_i \in X} \overline{R^-(x_i)} = \{(x_j, \mu_{N^{\mathrm{L}}(X,R)}(x_j)),\ j = 1,\ldots,n\},$$

where $\overline{R^-(x_i)} = \{(x_j, \mathrm{Neg}(r_{ij})),\ \forall x_j \in X\}$. $\mu_{N^{\mathrm{L}}(X,R)}$ is the characteristic membership function of $N^{\mathrm{L}}(X,R)$ that assigns a linguistic degree of "nondomination" to each alternative $x_j \in X$ with respect to R according to the next expression:

$$\mu_{N^{\mathrm{L}}(X,R)} : X \to S, \quad \mu_{N^{\mathrm{L}}(X,R)}(x_j) = \nabla(\mathrm{Neg}(r_{ij}),\ i = 1,\ldots,n).$$

Definition 10. The linguistic choice set of maximal alternatives for R in X is a fuzzy set in X given by

$$M^{\mathrm{L}}(X,R) = \nabla_{x_i \in X}(\mathrm{LI}^{\rightarrow}(R^-(x_i), R^+(x_i))) = \{(x_j, \mu_{M^{\mathrm{L}}(X,R)}(x_j)),\ j = 1,\ldots,n\},$$

where $\mathrm{LI}^{\rightarrow} : S^2 \to S$ denotes a linguistic implication operator. $\mu_{M^{\mathrm{L}}(X,R)}$ is the characteristic membership function of $M^{\mathrm{L}}(X,R)$ that assigns a linguistic degree of "maximality" to each alternative $x_j \in X$ with respect to R and a linguistic implication operator according to the next expression:

$$\mu_{M^{\mathrm{L}}(X,R)} : X \to S,\ \mu_{M^{\mathrm{L}}(X,R)}(x_j) = \nabla(\mathrm{LI}^{\rightarrow}(r_{ij}, r_{ji}),\ i = 1,\ldots,n).$$

A linguistic implication operator $\mathrm{LI}^{\rightarrow}$ works like a fuzzy one, but its expression domain is linguistic. We proposed some linguistic implication operators in Ref. [7]. For example, $\forall a, w \in S$ we have:

(1) Kleene–Dienes's linguistic implication function:

$$\mathrm{LI}_1^{\rightarrow}(w,a) = \max(\mathrm{Neg}(w), a).$$

(2) Gödel's linguistic implication function:

$$\mathrm{LI}_2^{\rightarrow}(w,a) = \begin{cases} s_T & \text{if } w \leqslant a, \\ a & \text{otherwise.} \end{cases}$$

(3) Fodor's linguistic implication function:

$$\mathrm{LI}_3^{\rightarrow}(w,a) = \begin{cases} s_T & \text{if } w \leqslant a, \\ \max(\mathrm{Neg}(w), a) & \text{otherwise.} \end{cases}$$

(4) Lukasiewicz's linguistic implication function:

$$\mathrm{LI}_4^{\rightarrow}(w,a) = \begin{cases} s_T & \text{if } w < a, \\ \mathrm{Neg}(w-a) & \text{otherwise,} \end{cases}$$

where $w - a = s_h \in S$ with $w = s_l, a = s_t$ and $l = t + h$.

Definition 11. The linguistic choice set of strictly greatest alternatives for R in X is a fuzzy set in X given by

$$S^{\mathrm{L}}(X,R) = \nabla_{x_i \in X}(\mathrm{LC}^{\rightarrow}(\overline{R^{-}(x_i)}, R^{+}(x_i))) = \{(x_j, \mu_{S^{\mathrm{L}}(X,R)}(x_j)),\ j = 1,\ldots,n\},$$

where $\mathrm{LC}^{\rightarrow} : S^2 \rightarrow S$ denotes a linguistic conjunction operator. $\mu_{S^{\mathrm{L}}(X,R)}$ is the characteristic membership function of $S^{\mathrm{L}}(X,R)$ that assigns a linguistic degree of "strict greatestness" to each alternative $x_j \in X$ with respect to R and a linguistic conjunction operator according to the next expression:

$$\mu_{M^{\mathrm{L}}(X,R)} : X \rightarrow S,\ \mu_{M^{\mathrm{L}}(X,R)}(x_j) = \nabla(\mathrm{LC}^{\rightarrow}(\mathrm{Neg}(r_{ij}), r_{ji}),\ i = 1,\ldots,n).$$

Similarly, a linguistic conjunction operator $\mathrm{LC}^{\rightarrow}$ works like a fuzzy one, but its expression domain is linguistic. We proposed some linguistic conjunction operators in Ref. [7]. For example, $\forall a, w \in S$ we have:

1. The classical Min linguistic conjunction function:

$$\mathrm{LC}_1^{\rightarrow}(w,a) = \min(w,a).$$

2. The nilpotent Min linguistic conjunction function:

$$\mathrm{LC}_2^{\rightarrow}(w,a) = \begin{cases} \min(w,a) & \text{if } w > \mathrm{Neg}(a), \\ s_0 & \text{otherwise.} \end{cases}$$

3. The weakest linguistic conjunction function:

$$\mathrm{LC}_3^{\rightarrow}(w,a) = \begin{cases} \min(w,a) & \text{if } \max(w,a) = s_T, \\ s_0 & \text{otherwise.} \end{cases}$$

We can also define the linguistic choice sets of nondominated, maximal and strictly greatest alternatives for R according to Definition 7.

Proposition 2.

1. $N^{\mathrm{L}}(X,R) = G^{\mathrm{L}}(X, \overline{R^{-1}})$,
2. $M^{\mathrm{L}}(X,R) = G^{\mathrm{L}}(X, \mathrm{LI}^{\rightarrow}(R^{-1}, R))$ *and*
3. $S^{\mathrm{L}}(X,R) = G^{\mathrm{L}}(X, \mathrm{LC}^{\rightarrow}(\overline{R^{-1}}, R)))$, *where* R^{-1} *is defined by* $R^{-1}(x_i,x_j) = r_{ji},\ \forall i,j,$ *and thus,* $\overline{R^{-1}(x_i,x_j)} = \mathrm{Neg}(r_{ji}),\ \forall i,j.$

In Section 4, we introduce the concept of linguistic choice function which generalizes the linguistic choice sets of alternatives, present different linguistic choice functions to characterize the above linguistic choice sets and also study some rationality properties of more important linguistic choice functions.

4. Linguistic choice functions

Definition 12. A linguistic choice function for R in X is a fuzzy set in X defined as

$$C(X,R) = \{(x_i, \mu_{C(X,R)}(x_i))\},$$

where $\mu_{C(X,R)} : S^n \to S$ is a linguistic membership function that assigns a linguistic choice degree to each alternative $x_i \in X$ with respect to R according to an expression.

Therefore, the linguistic choice sets of alternatives $G^{\mathrm{L}}(X,R)$, $N^{\mathrm{L}}(X,R)$, $M^{\mathrm{L}}(X,R)$ and $S^{\mathrm{L}}(X,R)$ are, of course, linguistic choice functions for R in X.

From Proposition 2 and following Switalski's studies [21] we have:

Definition 13. Let $C(X,R)$ be a linguistic choice function for R in X representing the choice set of alternatives $G^{\mathrm{L}}(X,R)$, then the generalized linguistic choice sets of C-nondominated, C-maximal and C-strictly greatest alternatives are defined as
1. $N^C(X,R) = C(X, \overline{R^{-1}})$.
2. $M^C(X,R) = C(X, \mathrm{LI}^{\rightarrow}(\underline{R^{-1}}, R))$ and
3. $S^C(X,R) = C(X, \mathrm{LC}^{\rightarrow}(\overline{R^{-1}}, R))$,

respectively.

If we assume a linguistic choice function C, such that for two linguistic preference relations R_1 and R_2 in X, if $R_1 \subset R_2$ then $C(X,R_1) \subset C(X,R_2)$, the following proposition is an inmediate consequence of Definition 13.

Proposition 3. *Let $C(X,R)$ be a linguistic choice function for R in X representing any $G^{\mathrm{L}}(X,R)$, then we have*:

1. $C(X,R) \subset M^C(X,R)$, *i.e., every greatest alternative is maximal.*
2. $N^C(X,R) \subset M^C(X,R)$, *i.e., every nondominated alternative is maximal.*
3. $C(X,R) = M^C(X,R)$ *if R is complete, i.e., every maximal alternative is greatest if R is complete.*
4. $N^C(X,R) = M^C(X,R)$ *if R is antisymmetric, i.e., every maximal alternative is nondominated if R is antisymmetric.*
5. $S^C(X,R) \subset C(X,R)$, *i.e., every strictly greatest alternative is greatest.*
6. $S^C(X,R) = N^C(X,R)$ *if R is complete, i.e., every strictly greatest alternative is nondominated if R is complete.*
7. $S^C(X,R) = C(X,R)$ *if R is antisymmetric, i.e., every strictly greatest alternative is greatest if R is antisymmetric.*

Proof. The demonstration is easily deduced from Ref. [21]. For all $s_i, s_j \in S$ it is satisfied that:
1. $s_i \leqslant \mathrm{LI}^{\rightarrow}(s_j, s_i)$.
2. $\mathrm{Neg}(s_j) \leqslant \mathrm{LI}^{\rightarrow}(s_j, s_i)$.
3. If $\max(s_i, s_j) = s_{\mathrm{T}}$ then $\mathrm{LI}^{\rightarrow}(s_j, s_i) = s_i$.
4. If $\min(s_i, s_j) = s_0$ then $\mathrm{LI}^{\rightarrow}(s_j, s_i) = \mathrm{Neg}(s_j)$.
5. $s_i \geqslant \mathrm{LC}^{\rightarrow}(s_j, s_i)$.
6. If $\max(s_i, s_j) = s_{\mathrm{T}}$ then $\mathrm{LC}^{\rightarrow}(\mathrm{Neg}(s_j), s_i) = \mathrm{Neg}(s_j)$.
7. If $\min(s_i, s_j) = s_0$ then $\mathrm{LC}^{\rightarrow}(\mathrm{Neg}(s_j), s_i) = s_i$.

Hence, if R is a linguistic preference relation in X then,

1. $R \subset \mathrm{LI}^{\rightarrow}(R^{-1}, R)$, and thus, $C(X,R) \subset M^C(X,R)$.
2. $\overline{R^{-1}} \subset \mathrm{LI}^{\rightarrow}(R^{-1}, R)$, and thus, $N^C(X,R) \subset M^C(X,R)$.
3. If R is complete then $R = \mathrm{LI}^{\rightarrow}(R^{-1}, R)$, and thus, $C(X,R) = M^C(X,R)$.
4. If R is antisymmetric then $\overline{R^{-1}} = \mathrm{LI}^{\rightarrow}(R^{-1}, R)$, and thus, $N^C(X,R) = M^C(X,R)$.
5. $\mathrm{LC}^{\rightarrow}(\overline{R^{-1}}, R) \subset R$, and thus, $S^C(X,R) \subset C(X,R)$.
6. If R is complete then $\overline{R^{-1}} = \mathrm{LC}^{\rightarrow}(\overline{R^{-1}}, R)$, and thus, $S^C(X,R) = N^C(X,R)$.
7. If R is antisymmetric then $R = \mathrm{LC}^{\rightarrow}(\overline{R^{-1}}, R)$, and thus, $S^C(X,R) = C(X,R)$.

4.1. Particular linguistic choice functions

Some particular cases of linguistic choice functions are the following ones:

(1) $C_{\min}(X,R) = \{(x_i, \mu_{C_{\min}(X,R)}(x_i)), \forall x_i \in X\}$,

$$\mu_{C_{\min}(X,R)}(x_i) = \min(r_{i1}, \ldots, r_{in}).$$

This linguistic choice function is a linguistic version of a fuzzy choice function proposed in Ref. [21].

(2) $C_{\phi_Q}(X,R) = \{(x_i, \mu_{C_{\phi_Q}(X,R)}(x_i)), \forall x_i \in X\}$,

$$\mu_{C_{\phi_Q}(X,R)}(x_i) = \phi_Q(r_{i1}, \ldots, r_{in}).$$

ϕ_Q is an aggregation operator of linguistic information guided by a nondecreasing fuzzy linguistic quantifier [25] $Q: [0,1] \rightarrow [0,1]$, called the LOWA operator [12] and which is a "nondecreasing" and "or-and" operator [9]. This linguistic choice functions was proposed in Ref. [8] and called *quantifier guided dominance linguistic choice function*. Clearly, it generalizes the above linguistic choice function due to the properties of the LOWA operator.

(3) $C_{Q^{\mathrm{L}}}(X,R) = \{(x_i, \mu_{C_{Q^{\mathrm{L}}}(X,R)}(x_i)), \forall x_i \in X\}$,

$$\mu_{C_{Q^{\mathrm{L}}}(X,R)}(x_i) = Q^{\mathrm{L}}\left(\frac{(r_{i1} > r_{1i}) + \cdots + (r_{in} > r_{ni})}{n-1}\right).$$

Q^{L} is a nondecreasing fuzzy linguistic quantifier valued linguistically in S [9]. $(r_{ij} > r_{ji})$ is a statement with value 1 if it is satisfied and 0 otherwise. This linguistic choice function was proposed in Ref. [9] and called *quantifier guided strict dominance linguistic choice function*.

(4) $C'_{\min}(X,R) = \{(x_i, \mu_{C'_{\min}(X,R)}(x_i)), \forall x_i \in X\}$,

$$\mu_{C'_{\min}(X,R)}(x_i) = \min(\mathrm{Neg}(r_{1i}), \ldots, \mathrm{Neg}(r_{ni})).$$

This linguistic choice function is a linguistic version of a fuzzy choice function proposed in Ref. [18].

(5) $C_{\min \circ \mathrm{LC}^{\rightarrow}}(X,R) = \{(x_i, \mu_{C_{\min \circ \mathrm{LC}^{\rightarrow}}(X,R)}(x_i)), \forall x_i \in X\}$,

$$\mu_{C_{\min \circ \mathrm{LC}^{\rightarrow}}(X,R)}(x_i) = \min(\mathrm{LC}^{\rightarrow}(\mathrm{Neg}(r_{1i}), r_{i1}), \ldots, \mathrm{LC}^{\rightarrow}(\mathrm{Neg}(r_{ni}), r_{in})).$$

Similarly, this linguistic choice function is a linguistic version of a fuzzy choice function proposed in Ref. [18].

The first three linguistic choice functions are particular cases of the linguistic choice set $G^{\mathrm{L}}(X,R)$, and thus, they assign degrees of greatestness to each alternative. The four and five ones are particular cases of $N^{\mathrm{L}}(X,R)$ and $S^{\mathrm{L}}(X,R)$, i.e., they assign degrees of nondomination and strict greatestness to each alternative, respectively.

The next linguistic choice functions are examples of the linguistic choice set $M^{\mathrm{L}}(X,R)$ defined using Lukasiewicz's linguistic implication function $\mathrm{LI}_4^{\rightarrow}$.

(1) $C_{\min \circ \mathrm{LI}_4^{\rightarrow}}(X,R) = \{(x_i, \mu_{C_{\min \circ \mathrm{LI}_4^{\rightarrow}}(X,R)}(x_i)), \forall x_i \in X\}$,

$\mu_{C_{\min \circ \mathrm{LI}_4^{\rightarrow}}(X,R)}(x_i) = \min(\mathrm{LI}_4^{\rightarrow}(r_{1i}, r_{i1}), \ldots, \mathrm{LI}_4^{\rightarrow}(r_{ni}, r_{in}))$.

This choice function, called *nondominance linguistic choice function*, was defined in Ref. [8] as a linguistic version of the Orlovsky's fuzzy choice set of nondominated alternatives [16].

(2) $C_{\phi_Q \circ \mathrm{LI}_4^{\rightarrow}}(X,R) = \{(x_i, \mu_{C_{\phi_Q \circ \mathrm{LI}_4^{\rightarrow}}(X,R)}(x_i)), \forall x_i \in X\}$,

$\mu_{C_{\phi_Q \circ \mathrm{LI}_4^{\rightarrow}}(X,R)}(x_i) = \phi_Q(\mathrm{LI}_4^{\rightarrow}(r_{1i}, r_{i1}), \ldots, \mathrm{LI}_4^{\rightarrow}(r_{ni}, r_{in}))$.

This choice function, called *quantifier guided nondominance linguistic choice function* was presented in Ref. [11] as a generalization of the above one.

In the following, we present three generalized linguistic choice functions which assign choice degrees in the senses of $N^{\mathrm{L}}(X,R)$, $S^{\mathrm{L}}(X,R)$ and $M^{\mathrm{L}}(X,R)$, respectively.

Definition 14. A generalized quantifier guided nondominated linguistic choice function is defined as $C'_{\phi_Q}(X,R) = \{(x_i, \mu_{C'_{\phi_Q}(X,R)}(x_i)), \forall x_i \in X\}$,

$$\mu_{C'_{\phi_Q}(X,R)}(x_i) = \phi_Q(\mathrm{Neg}(r_{1i}), \ldots, \mathrm{Neg}(r_{ni})).$$

Definition 15. A generalized quantifier guided strictly greatest linguistic choice function is defined as $C_{\phi_Q \circ \mathrm{LC}^{\rightarrow}}(X,R) = \{(x_i, \mu_{C_{\phi_Q \circ \mathrm{LC}^{\rightarrow}}(X,R)}(x_i)), \forall x_i \in X\}$,

$$\mu_{C_{\phi_Q \circ \mathrm{LC}^{\rightarrow}}(X,R)}(x_i) = \phi_Q(\mathrm{LC}^{\rightarrow}(\mathrm{Neg}(r_{1i}), r_{i1}), \ldots, \mathrm{LC}^{\rightarrow}(\mathrm{Neg}(r_{ni}), r_{in})).$$

Definition 16. A generalized quantifier guided maximal linguistic choice function is defined as $C_{\phi_Q \circ \mathrm{LI}^{\rightarrow}}(X,R) = \{(x_i, \mu_{C_{\phi_Q \circ \mathrm{LI}^{\rightarrow}}(X,R)}(x_i)), \forall x_i \in X\}$,

$$\mu_{C_{\phi_Q \circ \mathrm{LI}^{\rightarrow}}(X,R)}(x_i) = \phi_Q(\mathrm{LI}^{\rightarrow}(r_{1i}, r_{i1}), \ldots, \mathrm{LI}^{\rightarrow}(r_{ni}, r_{in})).$$

Remark 1. Summarizing, we have presented the following linguistic choice functions:
1. In the sense of the linguistic choice set $G^{\mathrm{L}}(X,R)$: $C_{\min}(X,R)$, $C_{\phi_Q}(X,R)$, $C_{Q^{\mathrm{L}}}(X,R)$.
2. In the sense of the linguistic choice set $N^{\mathrm{L}}(X,R)$: $C'_{\min}(X,R)$, $C'_{\phi_Q}(X,R)$.
3. In the sense of the linguistic choice set $S^{\mathrm{L}}(X,R)$,: $C_{\min \circ \mathrm{LC}^{\rightarrow}}(X,R)$, $C_{\phi_Q \circ \mathrm{LC}^{\rightarrow}}(X,R)$.
4. In the sense of the linguistic choice set $M^{\mathrm{L}}(X,R)$: $C_{\min \circ \mathrm{LI}_4^{\rightarrow}}(X,R)$, $C_{\phi_Q \circ \mathrm{LI}_4^{\rightarrow}}(X,R)$, $C_{\phi_Q \circ \mathrm{LI}^{\rightarrow}}(X,R)$.

From Proposition 3 the following results are an inmediate consequence.
1. $C_{\min}(X,R) \subset C_{\min \circ \mathrm{LI}_4^{\rightarrow}}(X,R)$ and $C_{\min}(X,R) \subset C_{\min \circ \mathrm{LI}^{\rightarrow}}(X,R)$.
2. $C'_{\min}(X,R) \subset C_{\phi_Q \circ \mathrm{LI}_4^{\rightarrow}}(X,R)$ and $C'_{\min}(X,R) \subset C_{\phi_Q \circ \mathrm{LI}^{\rightarrow}}(X,R)$.
3. $C'_{\phi_Q}(X,R) \subset C_{\phi_Q \circ \mathrm{LI}_4^{\rightarrow}}(X,R)$ and $C'_{\phi_Q}(X,R) \subset C_{\phi_Q \circ \mathrm{LI}^{\rightarrow}}(X,R)$.
4. $C_{\min \circ \mathrm{LC}^{\rightarrow}}(X,R) \subset C_{\min}(X,R)$ and $C_{\phi_Q \circ \mathrm{LC}^{\rightarrow}}(X,R) \subset C_{\phi_Q}(X,R)$.
5. $C_{\min \circ \mathrm{LC}^{\rightarrow}}(X,R) \subset C_{Q^{\mathrm{L}}}(X,R)$ and $C_{\phi_Q \circ \mathrm{LC}^{\rightarrow}}(X,R) \subset C_{Q^{\mathrm{L}}}(X,R)$.
6. $C_{\phi_Q}(X,R) \subset C_{\phi_Q \circ \mathrm{LI}_4^{\rightarrow}}(X,R)$ and $C_{\phi_Q}(X,R) \subset C_{\phi_Q \circ \mathrm{LI}^{\rightarrow}}(X,R)$.

Example 1. We show an example of above linguistic choice functions assuming the following linguistic preference relation R_1 defined in a set of three alternatives $X = \{x_1, x_2, x_3\}$ using the set of nine linguistic terms:

$$R_1 = \begin{bmatrix} C & C & I \\ \mathrm{MC} & C & \mathrm{EL} \\ C & C & C \end{bmatrix}.$$

Then, if Q and Q^{L} are the fuzzy linguistic quantifier "*As many as possible*" [9] we have the following choice sets of alternatives:

$C_{\min}(X,R_1) = \{(x_1,I),(x_2,\mathrm{MC}),(x_3,C)\}$. $C_{\phi_Q}(X,R_1) = \{(x_1,C),(x_2,\mathrm{EL}),(x_3,C)\}$.
$C'_{\min}(X,R_1) = \{(x_1,I),(x_2,I),(x_3,I)\}$. $C'_{\phi_Q}(X,R_1) = \{(x_1,I),(x_2,I),(x_3,\mathrm{EU})\}$.
$C_{\min\circ\mathrm{LI}_4^{\rightarrow}}(X,R_1) = \{(x_1,I),(x_2,\mathrm{MC}),(x_3,C)\}$. $C_{\phi_Q\circ\mathrm{LI}_4^{\rightarrow}}(X,R_1) = \{(x_1,C),(x_2,\mathrm{EL}),(x_3,C)\}$.
$C_{\min\circ\mathrm{LI}_2^{\rightarrow}}(X,R_1) = \{(x_1,I),(x_2,\mathrm{MC}),(x_3,C)\}$. $C_{\phi_Q\circ\mathrm{LI}_2^{\rightarrow}}(X,R_1) = \{(x_1,C),(x_2,\mathrm{EL}),(x_3,C)\}$.
$C_{\min\circ\mathrm{LC}_1^{\rightarrow}}(X,R_1) = \{(x_1,I),(x_2,I),(x_3,I)\}$. $C_{\phi_Q\circ\mathrm{LC}_1^{\rightarrow}}(X,R_1) = \{(x_1,I),(x_2,I),(x_3,\mathrm{EU})\}$.
$C_{Q^L}(X,R_1) = \{(x_1,\mathrm{IM}),(x_2,I),(x_3,C)\}$.

On the other hand, as R_1 is a complete linguistic relation the next relations are satisfied:
1. $C_{\min}(X,R_1) = C_{\min\circ\mathrm{LI}_4^{\rightarrow}}(X,R_1)$ and $C_{\min}(X,R_1) = C_{\min\circ\mathrm{LI}_2^{\rightarrow}}(X,R_1)$.
2. $C_{\phi_Q}(X,R_1) = C_{\phi_Q\circ\mathrm{LI}_4^{\rightarrow}}(X,R_1)$ and $C_{\phi_Q}(X,R_1) = C_{\phi_Q\circ\mathrm{LI}_2^{\rightarrow}}(X,R_1)$.
3. $C_{\min\circ\mathrm{LC}_1^{\rightarrow}}(X,R_1) = C'_{\min}(X,R_1)$.
4. $C_{\phi_Q\circ\mathrm{LC}_1^{\rightarrow}}(X,R_1) = C'_{\phi_Q}(X,R_1)$.

4.2. Rationality properties of the linguistic choice functions

The characteristic properties of a choice function represent the reasonable conditions or rationality properties required to be considered as adeaquate [18,20]. Some of them are classical: *heritage condition, concordance condition and independance of irrelevant alternatives*. They have been analyzed for fuzzy choice functions in Refs. [1,15,18]. We revisit them for the more important linguistic choice functions presented here with a view to show their consistent acting ways. In particular, we study the two following set of linguistic choice functions:
1. $\{C_{\min}(X,R),\ C'_{\min}(X,R),\ C_{\min\circ\mathrm{LI}^{\rightarrow}}(X,R),\ C_{\min\circ\mathrm{LC}^{\rightarrow}}(X,R)\}$ and
2. $\{C_{\phi_Q}(X,R),\ \ C'_{\phi_Q}(X,R),\ C_{\phi_Q\circ\mathrm{LI}^{\rightarrow}}(X,R),\ \ C_{\phi_Q\circ\mathrm{LC}^{\rightarrow}}(X,R)\}$.
They may be represented generically as $C_{\min}(X,H^q)$ and $C_{\phi_Q}(X,H^q)$, respectively, with $q = 1,\ldots,4$ and H^q a linguistic preference relation defined as

$$H^1 = R,\quad H^2 = R^{-1},\quad H^3 = \mathrm{LI}^{\rightarrow}(R^{-1},R),\quad H^4 = \mathrm{LI}^{\rightarrow}(\overline{R^{-1}},R).$$

4.2.1. Heritage property

This property analyzes how must be the behaviour of a choice function when the number of the components of the set of alternatives is changed. In particular, this condition formalizes the fact that if an alternative is good it must follow being good when the set of alternatives is increased. This idea is expressed as follows.

Proposition 4. *For $X' \subseteq X$, then*

$$\left(X' \bigcap C_{\min}(X,H^q)\right) \subset C_{\min}(X',H^q|X'),\ \forall q,$$

with $\mu_{H^q|X'}(x_i,x_j) = h^q_{ij}, \forall x_i,x_j \in X'$.

Proof. This property is a consequence of the following property of the linguistic t-norm min:

$$\min(a_1, \dots, a_k) \leqslant \min(a_1, \dots, a_{k-1}), \forall a_j \in S.$$

The next proposition is a consequence of above one.

Proposition 5. *For $X' \subseteq X$, if the* LOWA *operator ϕ_Q satisfies the next condition*:

$$\phi_Q(a_1, \dots, a_k) \leqslant \phi_Q(a_1, \dots, a_{k-1}), \forall a_j \in S \ (label\ set),$$

then

$$\left(X' \bigcap C_{\phi_Q}(X, H^q)\right) \subset C_{\phi_Q}(X', H^q|X'), \ \forall q.$$

Thus, the heritage property is satisfied by the linguistic choice functions $C_{\phi_Q}(X, H^q)$ when the LOWA operator ϕ_Q presents characteristics similar to the linguistic t-norm min. In practice, it is achieved for some particular fuzzy linguistic quantifiers, for example, "As many as possible" or "All" [7].

4.2.2. Concordance property

This property is formalized as follows.

Proposition 6. *For all $X_1, \ X_2 \subseteq X$ and for all $x_i \in X$ it is satisfied that*

$$\min(C_{\min}(X_1, H^q|X_1)(x_i), C_{\min}(X_2, H^q|X_2)(x_i)) \leqslant C_{\min}\left(X_1 \bigcup X_2, H^q|\left(X_1 \bigcup X_2\right)\right)(x_i), \ \forall q.$$

Proof. This proposition is an inmediate consequence of the following property of linguistic t-norm min:

$$\min(a_1, \dots, a_k) = \min(\min(a_1, \dots, a_r), \min(a_{r+1}, \dots, a_k)), \ \forall a_j \in S.$$

Then, for all $x_i \in X_1 \bigcap X_2$ it is obvious that

$$\min(C_{\min}(X_1, H^q|X_1)(x_i), C_{\min}(X_2, H^q|X_2)(x_i)) = C_{\min}\left(X_1 \bigcup X_2, H^q|\left(X_1 \bigcup X_2\right)\right)(x_i), \ \forall q.$$

On the other hand, for all $x_i \notin X_1 \bigcap X_2$ we have

$$\min(C_{\min}(X_1, H^q|X_1)(x_i), C_{\min}(X_2, H^q|X_2)(x_i)) = s_0.$$

Proposition 7. *For all $X_1, \ X_2 \subseteq X$ if the LOWA operator ϕ_Q satisfies the following conditions*:
1. $\phi_Q(a_1, \dots, a_k) = \phi_Q(\phi_Q(a_1, \dots, a_r), \phi_Q(a_{r+1}, \dots, a_k)), \ \forall a_j \in S$, *and*
2. $\phi_Q(s_0, a_j) = s_0, \ \forall a_j \in S$,

then for all $x_i \in X$ it is satisfied that

$$\phi_Q(C_{\phi_Q}(X_1, H^q|X_1)(x_i), C_{\phi_Q}(X_2, H^q|X_2)(x_i)) \leqslant C_{\phi_Q}\left(X_1 \bigcup X_2, H^q|\left(X_1 \bigcup X_2\right)\right)(x_i) \ \forall q.$$

Similarly, the satisfaction of the concordance property by the linguistic choice functions $C_{\phi_Q}(X, H^q)$ depends on the weights used in the aggregation of the LOWA operator, which allow it to simulate the behaviour of a linguistic t-norm min.

4.2.3. Independence of irrelevant alternatives

This property as the heritage property also analyzes how must be the behaviour of a choice function when the number of the components of the set of alternatives is changed. In particular, this condition formalizes the fact that if an alternative is better than another, that relation must be fulfilled when the set of alternatives is increased.

Lemma 1. *If H^q is a reflexive and negatively transitive linguistic preference relation defined in a finite set of alternatives X, then there is no cycle $x_{v_0}, x_{v_1}, \ldots, x_{v_{z-1}}, x_{v_z} = x_{v_0}$ with the property $h^q_{v_l v_{l+1}} < s_T,\ l = 0, 1, \ldots, z-1$.*

Proof. Suppose that there is a cycle $x_{v_0}, x_{v_1}, \ldots, x_{v_{z-1}}, x_{v_z} = x_{v_0}$ with the property $h^q_{v_l v_{l+1}} < s_T,\ l = 0, 1, \ldots, z-1$, then by the property negatively transitive,

$$s_T > \max(h^q_{v_0 v_1}, h^q_{v_1 v_0}, \ldots, h^q_{v_{z-1} v_z}) \geqslant h^q_{v_0 v_0} = s_T,$$

and we find a contradiction.

Proposition 8. *Let H^q be a reflexive and negatively transitive linguistic relation in X. If $X' \subseteq X$ such that $\forall x_i \in X'$ and $\forall x_j \in X \setminus X'$ it is satisfied that*

$$C_{\min}(X, H^q)(x_j) < C_{\min}(X, H^q)(x_i), \quad \forall q,$$

then

$$C_{\min}(X', H^q|X')(x_k) = C_{\min}(X, H^q)(x_k), \quad \forall x_k \in X',\ \forall q.$$

Proof. From Proposition 4, $\forall x_k \in X'$, we have

$$C_{\min}(X', H^q|X')(x_k) \geqslant C_{\min}(X, H^q)(x_k), \quad \forall q.$$

Now, we must prove the following:

$$C_{\min}(X', H^q|X')(x_k) \leqslant C_{\min}(X, H^q)(x_k) \quad \forall x_k \in X',\ \forall q.$$

Suppose that there is $x_i \in X'$ such that $C_{\min}(X', H^q|X')(x_i) > C_{\min}(X, H^q)(x_i)$, $\forall q$, i.e.,

$$\min_{x_p \in X'}(h^q_{ip}) > \min_{x_p \in X}(h^q_{ip}), \quad \forall q.$$

It means that exists an alternative $x_j \in X \setminus X'$ such that

$$h^q_{ij} < h^q_{it}\ \forall x_t \in X', \quad \forall q. \quad (*)$$

On the other hand, we know that $C_{\min}(X, H^q)(x_j) < C_{\min}(X, H^q)(x_i)$, and thus, there is a x_{v_1} such that

$$h^q_{j v_1} < C_{\min}(X, H^q)(x_i), \quad \forall q.$$

If $x_{v_1} \in X \setminus X'$, there is a x_{v_2} such that

$$h^q_{v_1 v_2} < C_{\min}(X, H^q)(x_i), \quad \forall q,$$

and so on. Since X is a finite set and according to Lemma 1 there is no cycle with the property $h^q_{v_l v_{l+1}} < s_T$. We construct a sequence $x_{v_0} = x_j, x_{v_1}, \ldots, x_{v_z}$ such that $x_{v_l} \in X \setminus X'$ for $l < z$, $x_{v_z} \in X'$, and $h^q_{v_l v_{l+1}} < C_{\min}(X, H^q)(x_i)$ for $0 \leqslant l \leqslant z-1$. As H^q is negatively transitive, then $\forall x_p \in X$,

$$h^q_{jv_z} \leqslant \max_{l=0,\dots,z-1}\{h^q_{v_l v_{l+1}}\} < C_{\min}(X,H^q)(x_i) \leqslant h^q_{ip}, \quad \forall q.$$

In particular, $h^q_{jv_z} < h^q_{iv_z}$, which, together with (*), implies

$$\max(h^q_{ij}, h^q_{jv_z}) < h^q_{iv_z},$$

which is a contradiction with the negatively transitive property, and thus, there is not a $x_i \in X'$ such that

$$C_{\min}(X', H^q|X')(x_i) > C_{\min}(X,H^q)(x_i).$$

This proposition cannot be extended to linguistic choice functions $C_{\phi_Q}(X,H^q)$ without very strong restrictions on the LOWA operator. Therefore, it is satisfied when we use LOWA operators that work exactly equal to the min operator.

Remark 2. We should point out that we have presented a set of linguistic choice functions that satisfy many of the characteristic properties of choice functions required to be considered as adequate. Therefore, they seem appropriate to guide the choice processes of alternatives in those decision situations where the preferences are modeled in terms of linguistic preference relations. In short, we have given a panel of choice tools which may be used as the basis for developing the choice processes in different linguistic decision contexts as multi-criteria, multi-persons and multi-states decision making.

In Section 5, we study how to obtain from the linguistic choice functions the selection sets of the alternatives which define the solution for a given problem.

5. Linguistic choice mechanisms

A linguistic choice mechanism is a choice process which allows us to obtain a solution set of alternatives using linguistic choice functions. If we observe Example 1 we can obtain a solution set of alternatives of each linguistic choice function choosing those alternatives with maximum linguistic choice degree. However, if we apply this policy with all linguistic choice functions we do not always find the same solution. For example:

$$C^*_{Q^L}(X,R^1) = \{x_3\} \neq C^*_{\phi_Q}(X,R^1) = \{x_1, x_3\}.$$

Thus, we distinguish two kinds of linguistic choice mechanisms:

1. *Simple linguistic choice mechanisms.* They use only one linguistic choice function to obtain the solution set of alternatives. Therefore, given a linguistic choice function for R in X, $C(X,R)$, this method obtains the solution as

$$C^*(X,R) = \{x_j \in X \mid \mu_{C(X,R)}(x_j) = \max_{x_i \in X} \mu_{C(X,R)}(x_i)\},$$

i.e., those alternatives with maximum linguistic choice degree. We should point out that $C^*(X,R)$ can have one or various alternatives and that different choice functions may give different solutions.

2. *Composite linguistic choice mechanisms.* They use various linguistic choice functions to obtain the solution set of alternatives. They are applied when the solution obtained by the application of a simple mechanisms is not enough precise or specific. Futhermore, they also are useful when different simple mechanisms provide very different solutions. Therefore, we can say that a composite mechanism develops a consensus process between different choice functions with a view to achieve more specific solutions. Then, given a set of linguistic choice functions for R in X, $\{C_1(X,R), \dots, C_k(X,R)\}$, a composite linguistic choice mechanism obtains the solution set of alternatives $C^*(X,R)$ by the combined application of all linguistic choice functions. Usually, the combined application can be done according to two different policies:

(a) *Conjunctive policy.* This policy consists of applying in a parallel way all simple choice mechanisms of each choice function [11], i.e., it obtains the total solution as the intersection of the partial solutions according to the following expression:

$$C^*(X,R) = \bigcap_{t=1}^{T} C_t^*(X,R).$$

We should point out the existence of a problem, i.e., when it is verified that $\bigcap_{t=1}^{T} C_t^*(X,R) = \emptyset$. In such a situation, it is necessary to apply another choice policy as the following.
(b) *Sequential policy.* This policy consists of applying each one of the simple choice mechanisms of each choice function in sequence according to an order established previously [8]. Therefore, suppose that we have T simple linguistic choice mechanisms, then the total solution, called K_T, is obtained according to the following expression:

$$K_T = C_T^*(K_{T-1},R), \quad K_{T-1} = C_{T-1}^*(K_{T-2},R), \ldots,\ K_2 = C_2^*(K_1,R),\ K_1 = C_1^*(X,R).$$

5.1. Consistent linguistic choice mechanisms

In the application of any choice mechanism in a collective decision making process we should take into account the possible presence of preference cycles in the final linguistic preference relation that represents the collective opinion. This is a problem that makes the decision analysis more difficult, and deriving a clear and consistent solution becomes a complex task. It is the major difficulty of the application of any choice mechanism and it is a consequence of the Arrow Impossibility Theorem [2].

A way to evaluate this problem is by means of the transitivity property, and more particularly using the max–min transitivity [10]. According to the Impossibility Theorem a fine analysis shows that in the aggregation of the individual preferences the transitivity requirement and the independance of context property are incompatible [17]. Hence, an attempt to avoid this difficulty is to adopt context dependant comparisons of alternatives in order to get some transitivity property and in this a way, to incorporate more consistency in the choice mechanisms. Among the various techniques developed in this direction, procedures based on the *covering concept* are realy interesting [4]. They allow transitive information to be extracted from conventional cyclic preference relations. In Ref. [17] the application of covering concept for classical fuzzy preference relations is presented. Here, we extend it to the linguistic preference relations.

Roughly speaking, an alternative x_i is said to cover another alternative x_j if x_i beats x_j when they are compared vis-á-vis other alternatives respectively. In Ref. [17] Perny showed that only there is a possible way for the covering concept to be extended to the fuzzy case, if transitivity property is required. He gave a procedure to obtain a transitive relation from an intransitive one, which is defined in a linguistic context as follows.

Definition 17. Let R be a intransitive linguistic preference relation in X, then we can build a transitive linguistic preference relation $\mathrm{CC}(R)$ in X, called linguistic covering relation of R, as follows:

$$\forall (x_i,x_j) \in X^2, \quad CC(R)(x_i,x_j) = \min\{\mathrm{FC}(R)(x_i,x_j), \mathrm{BC}(R)(x_i,x_j)\},$$

where $\mathrm{FC}(R)$ and $\mathrm{BC}(R)$ are the linguistic forward and backward covering relations of R defined as

$$\mathrm{FC}(R) = \begin{cases} s_T & \text{if } \forall x_h \in X,\ r_{jh} \leqslant r_{ih} \\ \min\limits_{\{x_h \in X | r_{jh} > r_{ih}\}} \{r_{ih}\} & \text{otherwise.} \end{cases}$$

and

$$BC(R) = \begin{cases} s_T & \text{if } \forall x_h \in X,\ r_{hi} \leqslant r_{hj} \\ \min\limits_{\{x_h \in X | r_{hi} > r_{hj}\}} \{r_{hj}\} & \text{otherwise,} \end{cases}$$

respectively.

Hence, we propose to use *consistent linguistic choice mechanisms,* that is, linguistic choice mechanisms that act on the transitive linguistic preference relation CC(R) of R to avoid the appearance of consistency problems in the solution set of alternatives.

Following in this reasoning line and with a view to get a enough precise and consistent solution set of alternatives we present the next *consistent complete linguistic choice mechanism:*

1. Obtain CC(R) from R and choose various simple linguistic choice mechanisms.
2. Apply a *conjunction linguistic choice mechanism* on CC(R).
3. If $C^*(X, \mathrm{CC}(R)) = \emptyset$, then apply a *sequential linguistic choice mechanism* on CC(R). Otherwise $C^*(X, \mathrm{CC}(R))$ is the solution.

In Section 5.2, we present an example of application of the proposed methods.

5.2. Example

Suppose an investment company, which wants to invest a sum of money in the best option. There is a panel with four possibles options where to invest the money:

- x_1 is a car company,
- x_2 is a food company,
- x_3 is a computer company,
- x_4 is an arms company.

The investment company has a group of four consultancy departments:

- d_1 is the risk analysis department,
- d_2 is the growth analysis department,
- d_3 is the social-political impact analysis department, and
- d_4 is the environmental impact analysis department.

In each department there is one expert with different importance degrees (c_i for the expert of the department d_i):

$$\{c_1 = s_4, c_2 = s_5, c_3 = s_2, c_4 = s_6\}.$$

Assume to express the preferences the set of nine labels presented in Section 2, that is,

$$S = \{s_8 = C, s_7 = \mathrm{EL}, s_6 = \mathrm{ML}, s_5 = \mathrm{MC}, s_4 = \mathrm{IM}, s_3 = \mathrm{SC}, s_2 = \mathrm{VLC}, s_1 = \mathrm{EU}, s_0 = I\}.$$

Then, consider that the experts provide their assessments on the option set by means of the following linguistic preference relations:

$$R^1 = \begin{bmatrix} C & SC & MC & VLC \\ MC & C & IM & IM \\ SC & IM & C & VLC \\ ML & IM & ML & C \end{bmatrix}, \quad R^2 = \begin{bmatrix} C & IM & IM & VLC \\ IM & C & MC & IM \\ IM & SC & C & VLC \\ ML & IM & ML & C \end{bmatrix},$$

$$R^3 = \begin{bmatrix} C & IM & MC & I \\ IM & C & ML & IM \\ SC & VLC & C & VLC \\ C & IM & ML & C \end{bmatrix}, \quad R^4 = \begin{bmatrix} C & IM & MC & SC \\ IM & C & IM & SC \\ SC & IM & C & VLC \\ MC & MC & ML & C \end{bmatrix}.$$

Using the linguistic quantifier Q "At least half" with the pair $(0.0, 0.5)$, and the corresponding LOWA operator ϕ_Q with $W = (0.5, 0.5, 0, 0)$, the collective linguistic preference relation is:

$$R^c = \begin{bmatrix} C & IM & MC & VLC \\ IM & C & ML & IM \\ SC & IM & C & VLC \\ EL & IM & ML & C \end{bmatrix}.$$

This relation is not transitive, e.g., we find that $R^c_{14} = \mathrm{VLC} \leqslant \min(R^c_{12} = \mathrm{IM}, R^c_{24} = \mathrm{IM})$. Then, in order to obtain the solution set of options we apply the consistent complete linguistic choice mechanism. We obtain the following transitive relation from R^c:

$$\mathrm{CC}(R^c) = \begin{bmatrix} C & VLC & MC & VLC \\ SC & C & ML & VLC \\ SC & VLC & C & VLC \\ EL & IM & ML & C \end{bmatrix}.$$

Applying the simple choice mechanisms based on the linguistic choice functions $C_{\min}$ and C'_{ϕ_Q} we have the following linguistic choice sets:

$$C_{\min}(X, \mathrm{CC}(R^c)) = \{(x_1, \mathrm{VLC}), (x_2, \mathrm{VLC}), (x_3, \mathrm{VLC}), (x_4, \mathrm{IM})\},$$

$$C'_{\phi_Q}(X, \mathrm{CC}(R^c)) = \{(x_1, \mathrm{MC}), (x_2, \mathrm{ML}), (x_3, \mathrm{SC}), (x_4, \mathrm{ML})\}.$$

Then, we the solution using the conjunctive policy, i.e,

$$C^*(X, \mathrm{CC}(R^c)) = C^*_1(X, \mathrm{CC}(R^c)) \bigcap C^*_2(X, \mathrm{CC}(R^c)) = \{x_4\} \bigcap \{x_2, x_4\} = x_4,$$

that is, the best option is to invest in the arms company.

6. Concluding remarks

Here, we briefly summarize the results presented and also point out some considerations about them.

In this paper, we have studied the problem of finding a solution set of alternatives when a final selection opinion is given by means of a linguistic preference relation. First, we have dealt with establishing the choice set of alternatives for a linguistic preference relation. We have defined various linguistic choice sets of alternatives and they have been characterized by means of the concept of linguistic choice function. A set of linguistic choice functions based on the linguistic conjunctive function min and the LOWA operator have been presented and analyzed showing their rationality properties. Then, we have dealt with obtaining a solution set of alternatives from the choice set of alternatives. We have proposed a general method to apply the linguistic choice functions. Depending on the precision required we have given two application mechanisms of linguistic choice functions to obtain the solution set of alternatives: simple (less precise) and composite (more precise) linguistic choice mechanisms. We have introduced the concept of linguistic covering relation with a view to eliminate the presence of possible incoherence problems in the solution set of alternatives when the linguistic preference relation is intransitive. So, we have proposed a consistent complete linguistic choice mechanism which allows us to obtain more precise and coherent solutions.

The presented study provides a set of adequate tools to develop choice processes in decision contexts where the preferences are expressed by means of the linguistic preference relations. An important aspect of the presented results is that the proposed linguistic choice function present some interesting and natural rationality properties. On the other hand, another interesting result is that the proposed composite linguistic choice mechanisms allow to obtain more precise and coherent solutions according to different choice functions, i.e., they allow us to achieve consensus solutions with respect to all choice functions. Therefore, the presented tools show a satisfactory behaviour and may be easily applied to guide the choice processes in different linguistic decision contexts, e.g., multi-criteria, multi-persons and multi-states decision making.

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