ELSEVIER

Fuzzy Sets and Systems 88 (1997) 31–49

FUZZY sets and systems

A rational consensus model in group decision making using linguistic assessments

F. Herrera*, E. Herrera-Viedma, J.L. Verdegay

Department of Computer Science and Artificial Intelligence, University of Granada, 18071 Granada, Spain

Received June 1995; revised January 1996

Abstract

A new consensus model for the consensus reaching process, in a linguistic framework, is presented in heterogeneous group decision making problems, called rational consensus model. It is guided by some linguistic consensus and linguistic consistency meaures. All the measures are calculated from a set of linguistic preference relations used to provide experts' opinions. This consensus model allows more rational consensus solutions to be obtained and thus, more human consistency to be incorporated in decision support systems. © 1997 Elsevier Science B.V.

Keywords: Group decision making; Linguistic modelling; Consensus degrees; Consistency

1. Introduction

Human beings are constantly making decisions in the real world. In many situations, making decisions depends on numerous factors and therefore, given the limitations of human ability, it is very difficult to deal with. In such a case, the use of computerized decision support systems may be very helpful in solving decision making problems. In these systems, the problem is how to introduce intelligence, i.e., how to incorporate human consistency in decision making models of decision support systems. This problem has been dealt with successfully by means of fuzzy-logic-based tools, obtaining interesting results in the different decision making models. A classification for all of them is shown in [19], according to the number of stages before the decision is reached. We are interested in one fuzzy model in single-stage decision making, i.e., a fuzzy multi-person decision making model applied to group decision theory.

A group decision making problem may be defined as a decision situation in which there are two or more experts (i) each of them characterized by his/her own perceptions, attitudes, motivations, and personalities, (ii) who recognize the existence of a common problem, and (iii) who attempt to reach a collective decision. When the experts' opinions are not considered with the same intensity, it is known as a *heterogeneous group decision making problem*, and in another case, it is known as a *homogeneous group decision making problem*. In this paper, we focus on the heterogeneous group decision making model.

In a classical fuzzy environment, a heterogeneous group decision problem is considered as follows. It is assumed that there is a finite set of alternatives $X = \{x_1, \dots, x_n\}$ as well as a finite set of experts $E = \{e_1, \dots, e_m\}$ with their respective *importance degrees* defined as a fuzzy subset, such that, $\mu_G(k) \in [0, 1]$

* Corresponding author.
E-mail: herrera,viedma,verdegay@robinson.ugr.es.

PII S0165-0114(96)00047-4

denotes the importance degree of expert e_k. Each expert $e_k \in E$ provides his/her opinions on X as a fuzzy preference relation $P^k \subset X \times X$, with $p_{ij}^k \in [0, 1]$ denoting the preference degree of the alternative x_i over x_j.

Usually, in fuzzy environments, a standard assumption to express experts' preferences p_{ij}^k is by using numerical values assessed in a unit interval $[0, 1]$. However, there are some decision problems where experts are not able to give exact numerical values to their preferences. In such cases, an alternative option considered has been the use of linguistic assessments, instead of numerical values to express preferences [6, 8, 10, 20, 25, 26, 28]. Then, according to the problem domain, an appropriate linguistic term set is chosen and used by experts to describe their preferences. On the other hand, there are some decision problems where some experts prefer expressing their preferences with numerical values and others with linguistic values. Therefore, from this point of view, a group decision problem can be presented in a numerical framework (classical fuzzy environment), or in a linguistic framework, or a numerical and linguistic framework, depending on the nature of the expert's preferences. In this paper, we shall work in a linguistic framework, we shall consider that experts' opinions are provided by means of linguistic preference relations and their respective importance degrees by means of linguistic terms.

In a group decision making situation there are basically two problems to solve:

(i) *alternatives selection problem*, i.e., how to obtain solution alternative(s) set, and

(ii) *consensus problem*, i.e., how to achieve the maximum consensus degree from a group of experts for a solution alternative(s) set when they have diverging opinions. Both problems have been studied involving a numerical framework in [17,18]. We have studied and proposed solutions, in a linguistic framework, to the problem (i) in [9–13,15] and, in a numerical and linguistic framework to the problem (ii) in [14]. Here, we shall focus on the consensus problem.

In a usual context, the consensus problem is solved by means of a *consensus reaching process* [3, 16, 20, 14]. This is viewed as a dynamic and iterative process where a moderator, via the exchange of information and rational arguments, tries to persuade the experts to alter their opinions. At each step, the degree of consensus existing among experts' opinions is measured by means of a *consensus measure*. The moderator uses this consensus measure to control the process. This is repeated until experts' opinions become sufficiently similar.

On the other hand, usually, a group of experts initially presents inconsistencies in their opinions, i.e., they are not perfectly coherent in their judgments about the alternative set. In such a case, a desirable objective is to find a way of removing the inconsistencies of experts' judgments before obtaining the consensus solution, since otherwise, there may be no selective consensus solution (e.g., if it incorporates all the alternatives) or it may be distorted (e.g., if it does not incorporate the best alternatives). To solve the problem, we have considered incorporating a *consistency measure* in the consensus reaching process, which indicates the consistency degree of each expert at each moment of the process and, may be used by the moderator, together with a consensus measure, to control the process, and thus reach a more rational consensus solution. This is shown in Fig. 1.

In short, on the basis of above ideas, here, we present a *rational consensus model* for heterogeneous groups of experts using our model presented in [14]. It is developed in a linguistic framework and guided by several linguistic consensus and linguistic consistency measures. In this way, we propose a new consensus model, that allows more rational consensus solutions to be obtained, i.e., less distorted consensus solutions due to inconsistencies in the experts' opinions.

In order to do so, the paper is structured as follows: there is an appendix (Appendix A) with the linguistic framework considered, which should be read by researchers who are unfamiliar with the subject; Section 2 presents the rational consensus model; Section 3 describes its consensus measuring process; Section 4 describes its consistency measuring process; and finally, Section 5 contains our conclusions.

2. Rational consensus model

As we said at the beginning, we are assuming a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ as well as a finite and heterogeneous set of experts $E = \{e_1, \ldots, e_m\}$. For each expert $e_k \in E$, we shall suppose a defined importance degree, linguistically assessed in the term set, S (defined in Appendix A), and $\mu_G(k) \in S$, from

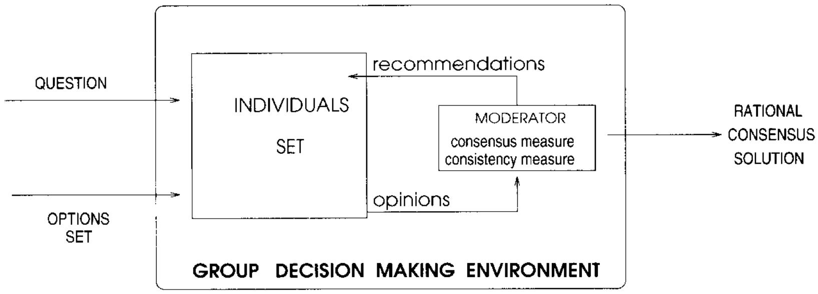

Fig. 1. Consensus reaching process guided by consensus and consistency measures.

s_0, standing for 'definitely irrelevant' and s_T, standing for 'definitely relevant', through all intermediate values. Then, the described model considers that each expert $e_k \in E$ provides his/her opinions on X as a linguistic preference relation, $P^k, \mu_{P^k} : X \times X \to S$, where $\mu_{P^k}(x_i, x_j) = p_{ij}^k \in S$ represents the linguistically assessed preference degree of the alternative x_i over x_j. Following [1], we assume that P^k is *soft-reciprocal* in the sense,

1. By definition $p_{ii}^k = s_0\ \forall x_i \in X$ (the minimum label in S).
2. If $p_{ij}^k \geqslant s_{T/2}$, then $p_{ji}^k \leqslant s_{T/2}$.

Condition 1 is a convection; if x_i is considered singly, no preference is assigned. Condition 2 seems plausible, because when $p_{ij}^k \geqslant s_{T/2}$, according to our definition of linguistic preference relations, given in Section 2, it is reasonable to think that the complementary preference p_{ji}^k should automatically be $\leqslant s_{T/2}$, since otherwise we would have a contradiction. In [1, 14], the second condition was established as $p_{ij}^k = \mathrm{Neg}(p_{ij}^k)$, but here we have relaxed it in order to allow more freedom in the experts' opinions.

On the other hand, following [4], *completeness* is required in order to ensure that all the experts consider the alternative set about which they are expressing their opinions, as feasible and comprehensive, where completeness is defined as $p_{ij}^k \geqslant \mathrm{Neg}(p_{ji}^k),\ \forall (x_i, x_j)$.

As was previously mentioned, consensus models are guided by consensus measures, i.e., measures of the degree of agreement between all the experts' opinions. For example, in [3, 16], in a numerical context, and in [20], in a linguistic context, consensus models are guided by numerical consensus measures. In [14], in a numerical and linguistic context, we proposed a new fuzzy logic based consensus model guided by several linguistic consensus measures. Now, we present a variation of our consensus model presented in [14] called as the '*rational consensus model*', which is developed in a linguistic environment and guided by several linguistic consensus and linguistic consistency measures. Its basic idea is that it develops a consensus reaching process which allows social consensus solutions and rational solutions to be obtained. It is guided by two types of measures: linguistic consensus and linguistic consistency measures.

Linguistic consensus measures are those described in [14], but calculated from a different perspective. In [14] we used an *average consensus policy*, i.e., each consensus value is obtained considering all the group of experts with a coincidence majority and their respective preference values. Here, we use a *strict consensus policy*, i.e., each consensus value is obtained by considering the preference value on the fact that the greatest degree of coincidence exists and its respective expert set, in other words, the largest group of experts and its respective preference value. They are applied in three acting levels: *level of preference, level of alternative, and level of preference relation.* These measures are:

1. *Consensus degrees.* Used to evaluate the current consensus stage, made up by three measures: *the preference linguistic consensus degree, the alternative linguistic consensus degree, and the relation linguistic consensus degree.*
2. *Linguistic distances.* Used to evaluate the individuals' distance from social opinions, made up of three measures: *the preference linguistic distance, the alternative linguistic distance, and the relation linguistic distance.*

There are two types of linguistic consistency measures, depending on the level of computation (expert or group of experts) and within these types, may be based either in qualitative aspects, i.e., obtained according to the intensity of the nature of the inconsistencies existing in the experts' opinions, or in quantitative aspects, i.e., obtained as a function of the quantity (in number) of inconsistencies existing amongst the experts' opinions:

1. *Individual consistency measures*, to evaluate the current consistency degree existing in the opinions of each expert. They are formed by two measures: *the quality-based individual consistency measure* and *the quantity-based individual consistency measure.*

2. *Collective consistency measures*, to evaluate the current consistency degree existing in the opinions of a group of experts. They are formed by two measures: *the quality-based collective consistency measure* and *the quantity-based collective consistency measure.*

All measures in the *rational consensus model* are obtained in two processes:

- *Consensus measuring process*, where *consensus measures* are calculated.
- *Consistency measuring process*, where *consistency measures* are calculated.

Both processes are developed in a parallel way at each step of the consensus reaching process until *acceptable consensus and consistency degrees* are achieved. The *rational consensus model* is reflected in Fig 2.

In the following sections, we analyze in detail each measuring process.

3. Consensus measuring process

This process follows the same scheme described in [14], i.e., it is developed in three phases:

1. *Counting process.* To count the individuals' opinions about preference values. From linguistic preference relations given by individuals, the number of individuals who are in agreement about the preference value of each alternative pair (x_i, x_j) is calculated and stored in the two *coincidence arrays.*

2. *Coincidence process.* To calculate the *coincidence degree*, i.e., the proportion of individuals who are in agreement in their preference values, and also to calculate *the consensus labels*, i.e., the majority opinion about preference values. This process is based on the idea of *coincidence* of experts and preference values. We consider that *coincidence* exists over a label assigned to a preference value, when more than one expert has chosen that label.

Using the above coincidence arrays, two relations are calculated:

1. *Labels consensus relation* (*LCR*), which contains the *consensus labels* about each preference, and

2. *Individuals consensus relation* (*ICR*), which contains the *coincidence degrees* of each preference.

3. *Computing process.* Finally, in this process the *consensus measures* about the aforementioned consensus relations are calculated in their respective level.

These processes are analyzed in detail in the following subsections.

3.1. Counting process

First, from the set of linguistic preference relations P^k, we define an array, V_{ij}, for the $T + 1$ possible labels that can be assigned as preference value. Each component $V_{ij}[s_t]$, $i, j = 1, \ldots, n$, $t = 0, \ldots, T$, is a set of the experts' identification numbers, who selected the value s_t as a preference value of the pair (x_i, x_j). Each V_{ij} is calculated according to this expression, $V_{ij}[s_t] = \{k \mid p_{ij}^k = s_t,\ k = 1, \ldots, m\}, \forall s_t \in S$.

Now, we define a pair of arrays, called *coincidence arrays*, to store information referring to the number of experts and their respective importance degrees:

- The first, symbolized as V_{ij}^C, and called *individuals coincidence array*, contains in each position, s_t, the number of experts, which coincides when assigning the label s_t as the preference value. The components of this array are obtained as, $V_{ij}^C[s_t] = \sharp(V_{ij}[s_t]), \forall s_t \in S$, where $\sharp$ stands for the cardinal.
- The second, symbolized as V_{ij}^G, and called *degrees coincidence array*, contains in each position s_t the average label of the experts' importance degrees, which coincides when assigning the label s_t as the preference value.

$$V_{ij}^G[s_t] = \begin{cases} \phi_{Q^1}(\mu_G(z_1), \mu_G(z_2), \ldots, \mu_G(z_q)) \\ \qquad \text{if } V_{ij}^C[s_t] > 1,\ z_k \in V_{ij}[s_t],\ q = V_{ij}^C[s_t], \\ s_0 \quad \text{otherwise,} \end{cases}$$

where ϕ_{Q^1} is the LOWA operator whose weights are calculated using the quantifier Q^1.

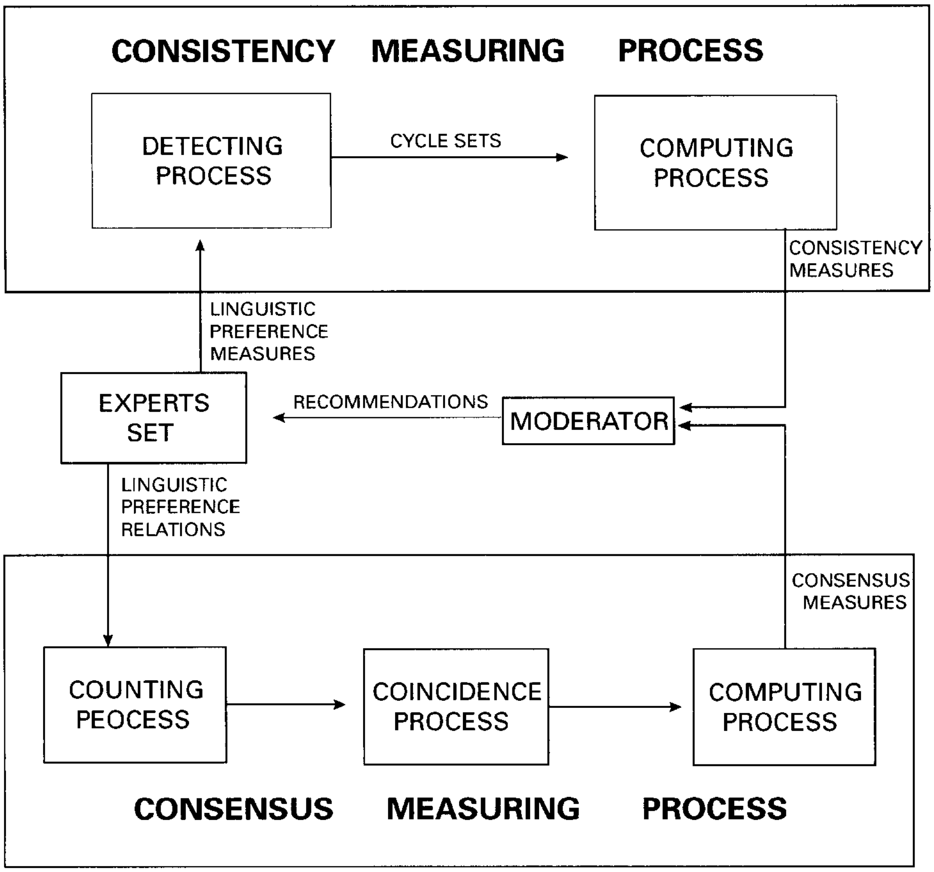

Fig. 2. Rational consensus model under linguistic assessments.

The linguistic quantifier Q^1 must be chosen properly to obtain a well-proportioned average importance degree.

3.2. Coincidence process

From the coincidence arrays, (V_{ij}^C, V_{ij}^G), the consensus relations, LCR and ICR are calculated under a *strict consensus policy*, which obtains consensus values (labels or the number of experts) as the maximum values of the *coincidence arrays* values.

1. *Labels consensus relation* (*LCR*). Each element (i,j) in the *labels consensus relation*, denoted by LCR_{ij}, represents the consensus label on the preference of each alternative pair (x_i,x_j). It is obtained as the linguistic index s_t of the maximum component $V_{ij}^C[s_t]$, such that, $V_{ij}^C[s_t] > 1$. That is, the linguistic label s_t, which has been selected the most by experts to evaluate the preference of the alternative pair (x_i,x_j). When there are several maximum labels, we choose the label chosen by experts with the highest average importance degree.

Before calculating LCR, we define the following parameter n_{ij}, which contains the maximum number of experts, who choose a given label, s_t to evaluate each alternative pair (x_i,x_j), $n_{ij} = \mathrm{MAX}_{s_t \in S}\{V_{ij}^C[s_t]\}$. Then we define label sets M_{ij}, for each alternative pair (x_i,x_j),

$$M_{ij} = \{s_y \mid V_{ij}^C[s_y] = n_{ij}, s_y \in S\},$$

which contain linguistic labels which have been chosen by the maximum number of experts n_{ij} to evaluate the preference of the respective alternative pair. Then if we call the label s_{ij} such that,

$$V_{ij}^G(s_{ij}) = \mathrm{MAX}_{s_y \in M_{ij}}\{V_{ij}^G(s_y)\},$$

we calculate each LCR_{ij}, according to the following expression,

$$\mathrm{LCR}_{ij}^s = \begin{cases} s_{ij} & \text{if } V_{ij}^C[s_{ij}] > 1, \\ \text{Undefined} & \text{otherwise.} \end{cases}$$

Note that value *undefined* means non-existence of coincidence on the label assigned according to the preference for a given alternative pair, i.e., its experts' coincidence array has all the components, $V_{ij}^C[s_t] \leqslant 1$.

2. *Individuals consensus relation* (*ICR*). Each element (i,j) of the *individuals consensus relation*, denoted by ICR_{ij}, represents the proportional number of experts whose preference value have been used to calculate the consensus label LCR_{ij}. Since we are interested in knowing the experts' importance degrees, we define two components for each ICR_{ij}. The first ICR_{ij}^1, containing the proportional number of experts, and the second ICR_{ij}^2, containing their respective *average importance degree*. Each component of ICR_{ij} is obtained as follows,

$$\mathrm{ICR}_{ij}^1 = \begin{cases} V_{ij}^C[s_{ij}]/m & \text{if } V_{ij}^C[s_{ij}] > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{ICR}_{ij}^2 = \begin{cases} V_{ij}^G[s_{ij}] & \text{if } V_{ij}^C[s_{ij}] > 1, \\ s_0 & \text{otherwise.} \end{cases}$$

3.3. *Computing process*

This process constitutes the last step in the consensus measuring process, in which the linguistic consensus measures are calculated. As mentioned earlier, there are two types of consensus measures:

- *Linguistic consensus degrees*. Used to evaluate current consensus existing among experts, and therefore the distance to the *ideal maximum consensus* (s_T). This type of measure helps the moderator to decide on the necessity to continue the consensus reaching process. Three linguistic consensus degrees are defined: *the preference linguistic consensus degree, the alternative linguistic consensus degree, and the relation linguistic consensus degree.*
- *Linguistic distances*. Used to evaluate how far the experts' opinions are from current consensus labels. This type of measure helps the moderator to identify which experts are furthest from the current majority consensus labels, and in which preferences the distance exists. Three linguistic distances are defined: *preference linguistic distance, alternative linguistic distance, and relation linguistic distance.*

The three measures for both types are calculated by distinguishing between three levels of computation: (i) *Level of the preference*; (ii) *Level of the alternative*; (iii) *Level of the preference relation.*

We obtain one measure of each type in its respective level. We calculate the *linguistic consensus degrees* using: (i) *Quantifier*, Q^2, *to represent the concept of fuzzy majority*, and (ii) *Individuals consensus relation*, ICR. We calculate the *linguistic distances* using: (i) *Preference relations of experts*, (ii) *LOWA operator*, (iii) *Quantifier* Q^1, *to represent the concept of fuzzy majority*, and (iv) *Labels consensus relation*, LCR. The computing process is shown in Fig. 3.

Next, we define each linguistic consensus measure in its respective level, by means of the aforementioned elements.

3.3.1. *Linguistic consensus degrees*

Before defining each degree, we introduce the concept of *consensus importance* over preference of pair (x_i,x_j), abbreviated by $\mu_I(x_{ij})$, defined as $\mu_I(x_{ij}) = ICR_{ij}^2$, representing the *importance of the consensus degree* achieved over each preference value. Futhermore, we use the linguistic valued quantifier, Q^2, which represents a *linguistic fuzzy majority of consensus.*

1. *Level* 1: *Preference linguistic consensus degree*

This degree is defined on the labels assigned to the preference of each pair (x_i,x_j), and it is denoted by PCR_{ij}. It indicates the consensus degree existing among all the m preference values attributed by the m experts to a specific preference. If we call PCR to the relation of all PCR_{ij}, then PCR is calculated as follows,

$$\mathrm{PCR}_{ij} = Q^2(\mathrm{ICR}_{ij}^1) \wedge \mu_I(x_{ij}),$$
$$i,j = 1,\ldots,n, \text{ and } i \neq j.$$

Therefore, in this model we always require the following condition $L = S$, i.e., the term set used by Q^2 must be equal to the one used by the group of experts to express their preferences.

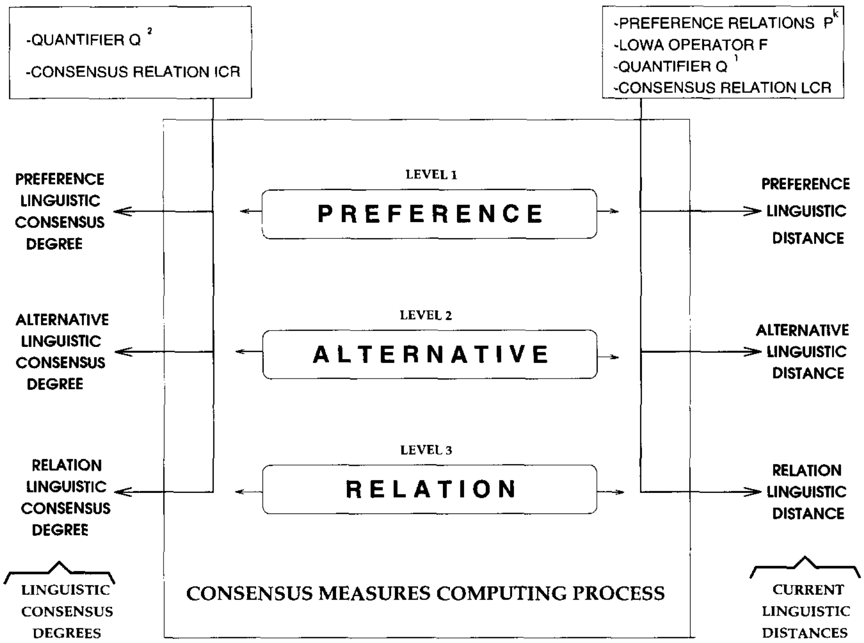

Fig. 3. Consensus computing process.

2. *Level* 2: *Alternative linguistic consensus degree*
This degree is defined on the label set assigned to all the preferences of one alternative x_i, and it is denoted by PCR_i. It allows us to measure the consensus existing over all the alternative pairs where a given alternative is present. It is calculated as

$$\mathrm{PCR}_i = \phi_{Q^1}(\mathrm{PCR}_{ij},\ j = 1,\dots,n,\ i \neq j),\quad i = 1,\dots,n.$$

3. *Level* 3: *Relation linguistic consensus degree*
This degree is defined on the preference relations of experts' opinions, and it is denoted by RC. It evaluates the social consensus, that is, the current consensus existing among all the experts about all the preferences. This is calculated as follows,

$$\mathrm{RC} = \phi_{Q^1}(\mathrm{PCR}_{ij},\ i,j = 1,\dots,n,\ \text{and } i \neq j).$$

3.3.2. Linguistic distances

The *linguistic distances* are defined similarly, by distinguishing between three acting levels, and using the above cited concepts. The idea is based on the evaluation of the approximation among experts' opinions and the current consensus labels of each preference.

1. *Level* 1: *Preference linguistic distance*
This distance is defined about the consensus label of the preference of each pair (x_i,x_j). It measures the distance between the opinions of an expert k about one preference and its respective consensus label. This is denoted by D^k_{ij}, and obtained as:

$$D^k_{ij} = \begin{cases} p^k_{ij} - \mathrm{LCR}_{ij} & \text{if } p^k_{ij} > \mathrm{LCR}_{ij}, \\ \mathrm{LCR}_{ij} - p^k_{ij} & \text{if } \mathrm{LCR}_{ij} \geqslant p^k_{ij},\ i \neq j \\ s_T & \text{otherwise}, \end{cases}$$

with $i,j = 1,\dots,n$, and $k = 1,\dots,m$ and, where if $p^k_{ij} = s_t$ and $\mathrm{LCR}_{ij} = s_v$ then $p^k_{ij} - \mathrm{LCR}_{ij}$ is defined as s_w, such that, $w = t - v$.

2. *Level* 2: *Alternative linguistic distance*
This distance is defined about the consensus labels of the preferences of one alternative x_i. It measures the distance between the preference values of an expert k about an alternative and its respective consensus labels. It is denoted by D^k_i, and obtained as follows

$$D^k_i = \phi_{Q^1}(D^k_{ij},\ j = 1,\dots,n,\ j \neq i),\quad k = 1,\dots,m, i = 1,\dots,n.$$

3. *Level* 3: *Relation linguistic distance*

This distance is defined about the consensus labels of group preference relation, LCR. It measures the distance between the preference values of an expert k over all alternatives and their respective *consensus labels*. It is denoted by D_R^k and obtained as follows,

$$D_R^k = \phi_{Q^1}(D_{ij}^k,\ i,j = 1,\ldots,n,\ j \neq i), \quad k = 1,\ldots,m.$$

In short, the main feature of the process described is of being very complete, because its measures allow the moderator to have plentiful information about the current consensus stage. In a direct way: information about the *consensus degree* by means of the *linguistic consensus degrees*, information about the *consensus labels* in every preference with the *label consensus relation*, and the behavior of the individuals during the consensus process, managing the *linguistic distances*. In an indirect way: information about the individuals, who are less in agreement, and in which preference this occurs, or information about the preferences where the agreement is high.

Below, we show the use of this process in one step of the consensus formation process, with a theoretical but clear example.

3.4. Application example of consensus measuring process

To illustrate the consensus reaching process proposed, from a practical point of view, consider the following nine linguistic label sets with their respective associated semantic, [2]:

C	*Certain*	$(1, 1, 0, 0)$
EL	*Extremely_likely*	$(0.98, 0.99, 0.05, 0.01)$
ML	*Most_likely*	$(0.78, 0.92, 0.06, 0.05)$
MC	*Meaningful_chance*	$(0.63, 0.80, 0.05, 0.06)$
IM	*It_may*	$(0.41, 0.58, 0.09, 0.07)$
SC	*Small_chance*	$(0.22, 0.36, 0.05, 0.06)$
VLC	*Very_low_chance*	$(0.1, 0.18, 0.06, 0.05)$
EU	*Extremely_unlikely*	$(0.01, 0.02, 0.01, 0.05)$
I	*Impossible*	$(0, 0, 0, 0)$

represented graphically in Fig. 4.

Let us consider four individuals, whose linguistic preferences, using the above label set are:

$$P^1 = \begin{bmatrix} - & SC & EL & VLC \\ MC & - & ML & EL \\ SC & SC & - & VLC \\ EL & IM & ML & - \end{bmatrix},$$

$$P^2 = \begin{bmatrix} - & MC & IM & VLC \\ IM & - & ML & IM \\ IM & SC & - & VLC \\ ML & MC & EL & - \end{bmatrix},$$

$$P^3 = \begin{bmatrix} - & EL & C & I \\ IM & - & MC & SC \\ EU & IM & - & VLC \\ C & EL & ML & - \end{bmatrix},$$

$$P^4 = \begin{bmatrix} - & SC & MC & SC \\ EL & - & IM & SC \\ IM & ML & - & VLC \\ C & MC & C & - \end{bmatrix},$$

respectively, and, whose respective linguistic importance degrees are:

$$\mu_G(1) = EL, \qquad \mu_G(2) = C,$$
$$\mu_G(3) = SC, \qquad \mu_G(4) = EU.$$

We shall use the linguistic quantifier Q = 'At least half' with the pair $(0.0, 0.5)$ for the process with its two versions, the numerical and linguistic values.

Then, some examples of components of *coincidence vectors* obtained in the counting process are:

$$V_{13}[MC] = \{4\}, \qquad V_{23}[C] = \{\emptyset\},$$
$$V_{24}[SC] = \{3,4\}, \qquad V_{41}[C] = \{3,4\},$$

with their respective components $(V_{ij}^C[s_t], V_{ij}^G[s_t])$:

$$V_{13}^C[MC] = 1, \qquad V_{23}^C[C] = 0,$$
$$V_{24}^C[SC] = 2, \qquad V_{41}^C[C] = 2,$$
$$V_{13}^G[MC] = I, \qquad V_{23}^G[C] = I,$$
$$V_{24}^G[SC] = SC, \qquad V_{41}^G[C] = SC.$$

In the coincidence process are obtained the relations:

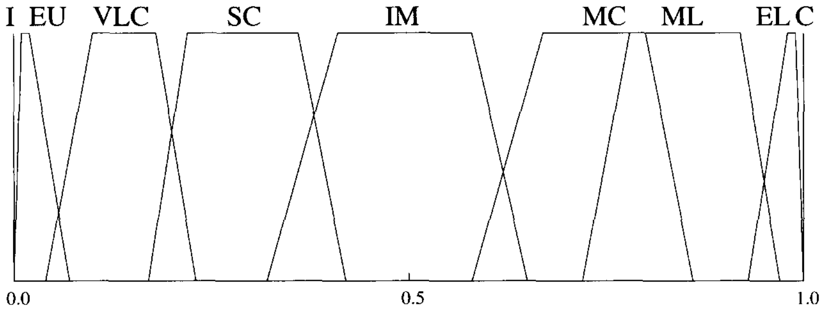

Fig. 4. Distribution of the nine linguistic labels.

Individuals consensus relations (ICR^1, ICR^2):

$$\mathrm{ICR}^1 = \begin{bmatrix} - & 0.5 & 0 & 0.5 \\ 0.5 & - & 0.5 & 0.5 \\ 0.5 & 0.5 & - & 1 \\ 0.5 & 0.5 & 0.5 & - \end{bmatrix},$$

$$\mathrm{ICR}^2 = \begin{bmatrix} - & EL & I & C \\ C & - & C & SC \\ C & C & - & C \\ SC & C & EL & - \end{bmatrix}.$$

Labels consensus relation (LCR):

$$\mathrm{LCR} = \begin{bmatrix} - & SC & ? & VLC \\ IM & - & ML & SC \\ IM & SC & - & VLC \\ C & MC & ML & - \end{bmatrix}$$

Note that the symbol ? of LCR indicates an undefined value in LCR_{13} because there is no label value with a coincidence value greater than 1.

From these *consensus relations*, we obtain the following *linguistic consensus degrees* and *linguistic distances*:

A. *Consensus degrees*

A.1. *Level of preference*. The *preference linguistic consensus degrees* are:

$$\mathrm{PCR} = \begin{bmatrix} - & EL & I & C \\ C & - & C & SC \\ EL & C & - & C \\ SC & C & EL & - \end{bmatrix}.$$

As may be observed, there are seven preferences where the consensus degree is total, according to the fuzzy majority of consensus established by the quantifier 'At least half'.

A.2. *Level of alternative*. The *alternative linguistic consensus degrees*, $\{\mathrm{PCR}_i\}$ are:

$$\mathrm{PCR}_1 = C, \qquad \mathrm{PCR}_2 = C,$$
$$\mathrm{PCR}_3 = C, \qquad \mathrm{PCR}_4 = C.$$

A.3. *Level of relation*. The *relation linguistic consensus degree* RC is

$$\mathrm{RC} = C.$$

Remarks. According to the concept of *linguistic fuzzy majority of consensus* introduced by the linguistically valued quantifier, Q, we obtain some conclusions: (i) *social consensus degree* is total, (ii) there is no consensus on the preference value of the pair (x_1,x_3), (iii) a great consensus degree exists on the majority of preference values, and (iv) all the alternatives present a *total consensus degree*, however, we can observe that consensus degrees over the preference values of alternative x_1, although high, are the smallest ones.

B. *The linguistic distances*. The *linguistic distances* of each expert e_k from *social consensus labels*, with $k = 1,\ldots,4$, are:

B.1. *Level of preference*. The *preference linguistic distances* are:

$$D^1 = \begin{bmatrix} - & I & C & I \\ EU & - & I & IM \\ EU & I & - & I \\ EU & EU & I & - \end{bmatrix},$$

$$D^2 = \begin{bmatrix} - & EU & C & I \\ I & - & I & EU \\ EU & I & - & I \\ VLC & I & EU & - \end{bmatrix},$$

$$D^3 = \begin{bmatrix} - & IM & C & VLC \\ I & - & EU & I \\ SC & EU & - & I \\ I & VLC & I & - \end{bmatrix},$$

$$D^4 = \begin{bmatrix} - & I & C & EU \\ SC & - & VLC & I \\ I & SC & - & I \\ I & I & VLC & - \end{bmatrix}.$$

B.2. *Level of alternative.* The *alternative linguistic distances* with Q^1 are:

Expert 1: $D_1^1 = MC$, $D_2^1 = SC$,
$D_3^1 = EU$, $D_4^1 = EU$,

Expert 2: $D_1^2 = ML$, $D_2^2 = EU$,
$D_3^2 = I$, $D_4^2 = VLC$,

Expert 3: $D_1^3 = EL$, $D_2^3 = EU$,
$D_3^3 = VLC$, $D_4^3 = EU$,

Expert 4: $D_1^4 = ML$, $D_2^4 = SC$,
$D_3^4 = VLC$, $D_4^4 = EU$.

B.3. *Level of relation.* The *relation linguistic distances* using Q^1 are:

$D_R^1 = SC$, $D_R^2 = VLC$,

$D_R^3 = SC$, $D_R^4 = SC$.

Remarks. We can draw some conclusions: (i) second expert presents less distance from the *current social cosensus stage*, (ii) all individuals are in disagreement on the current preference value of the pair (x_1, x_3), we must remember that LCR_{13}=?, and (iii) in the preference of alternative x_1 there is more disagreement and, in the preference for alternative x_4 there is less disagreement.

4. Consistency measuring process

As mentioned earlier, an important aspect of the theory of group decision making is the *problem of consistency or rationality of the group of experts*. Clearly, this problem itself includes two problems:

(i) when an expert, considered individually, is said to be rational, and

(ii) when a whole group of experts are considered rational.

In both cases, trying to give a full and definitive mathematical formalization of the general idea of *rationality* may be too abstract and complex. However, if the problem of rationality definition is focused from a point of view of the expert's opinions, that is, if the problem is analyzed according to the preferences expressed by experts, the problem may be more or less mathematically characterizable [5].

In a crisp context, where every expert expresses his/her opinions about pairs of alternatives of X by means of a crisp binary preference relation, R, the concept of *consistency* has traditionally been explained in terms of *acyclicity* [24], i.e., that the binary relation presents no sequence $x_1, x_2, \ldots, x_k$ (a 'cycle', being $x_{k+1} = x_1$) with $x_j \; R \; x_{j+1} \; \forall j = 1, \ldots, k$. On the other hand, in a fuzzy context, where every expert expresses his/her opinions by means of a fuzzy preference relation P, a well known standard assumption to characterize consistency is *max–min transitivity* [32]. Then, in both cases, an expert either is or is not considered consistent if his/her respective preference relaton either is or is not *acyclicity* (*max–min transitive*, respectively), and thus, in this sense, *consistency* is a crisp property. However, according to Montero [21, 22], we assume that the *consistency of experts* is clearly a fuzzy concept, since one expert's opinions can be considered more consistent than another expert's opinions. Therefore, *consistency* can be viewed as a fuzzy set defined by an appropriate membership function, called *fuzzy rationality measure*, which assigns to each expert a consistency value (degree) between 0 (absolute inconsistency) and 1 (absolute consistency), thereby obtaining a fuzzy classification of experts. In this sense, Montero provides, in [21, 22], a fuzzy rationality measure based in a particular weighted sum of all acyclicity paths and, Cutello and Montero propose in [5] an axiomatic definition that any explicitly consistent fuzzy rationality measure must satisfy. Here, working with linguistic preference relations, we propose two definitions of *linguistic consistency measures* in order to measure consistency of an expert, called *individual consistency measure*, and consistency of group of experts, called *group consistency measure*. Their definition is based on the *acyclicity idea* of Sen [24] and, following the definition of *fuzzy rationality* given by Montero in [21, 22], but in a linguistic context.

The measures are calculated in the consistency measuring process in two phases,

1. *Detecting process*, from linguistic preference relations given by experts, the inconsistent preference cycle sets considered in each relation are detected and obtained.

2. *Computing process*, from the aforementioned cycle sets detected for each expert, linguistic consistency measures are calculated.

Below, we analyze each process.

4.1. Detecting process

The aim of this process is to detect an inconsistent preference cycle set derived from each expert's preference relation. To do so, the cycles are not detected directly from each initial relation, but from each respective *strict relation*. The use of a strict relation is clearer observing real preference value existing among alternatives. Thus, a strict binary relation is defined for each binary relation and, inconsistent preference cycle sets are obtained therefrom. Each strict relation is obtained in Orlovski's sense [23] as follows.

Definition. Let P be a complete and soft reciprocal linguistic preference relation on $X = \{x_1, \ldots, x_n\}$ assessed in the term set S, $\mu_P : X \times X \to S$, where $\mu_P(x_i, x_j) = p_{ij}$. Then, $P^s = (p^s_{ij})$ is a *strict linguistic preference relation* assessed in the term set, $S' = S \cup \{\emptyset\}$, $\mu_{P^s} : X \times X \to S'$, where $\mu_{P^s}(x_i, x_j) = p^s_{ij}$, such that, $p^s_{ij} = \emptyset$ if $p_{ij} < p_{ji}$, or $p^s_{ij} = s_k \in S$ if $p_{ij} \geqslant p_{ji}$ with $p_{ij} = s_l$, $p_{ji} = s_t$ and $l = t + k$.

Therefore, working with strict preference relations, an alternative pair (x_i, x_j) can present any of these three basic relations:

1. *Preference relation* (R): x_i preferred to x_j, i.e., $x_i \; R \; x_j \Leftrightarrow p^s_{ij} > s_0$.

2. *No preference relation* (NR): x_i not preferred to x_j, i.e., $x_i \; NR \; x_j \Leftrightarrow p^s_{ij} = \emptyset$.

3. *Indifference relation* (I): x_i indifferent to x_j, i.e., $x_i \; I \; x_j \Leftrightarrow p^s_{ij} = s_0$.

Inverse relations of each relation (R, NR, I) are defined as: (1) $R^{-1} = NR$, (2) $NR^{-1} = R$, and (3) $I^{-1} = I$.

Observing P^s, it is clear that given a chain $x_1 - x_2 - \cdots - x_k - x_1$ of $k \geqslant 3$ distinct alternatives will be an inconsistent preference cycle, if and only if $x_1 R_1 x_2 R_2 \ldots x_k R_k x_1$, where either

Case 1: $R_h \in \{R, I\}$ $\forall h = 1, 2, \ldots, k$ and $NR \in \{R_h : h = 1, 2, \ldots, k\}$; or

Case 2: $R_h \in \{NR, I\}$ $\forall h = 1, 2, \ldots, k$ and $R \in \{R_h : h = 1, 2, \ldots, k\}$.

In other words, a chain will be an inconsistent preference cycle in each case, if R_h is either R (NR, respectively) or I, but having at least one R (NR, respectively).

Lemma. *Let P^s be a strict linguistic preference relation associated to a linguistic preference relation P, if $x_1 R_1 x_2 R_2 \ldots x_k R_k x_1$ is an inconsistent preference cycle of P^s, according to any case* (1 *or* 2), *then $x_1 R_k^{-1} x_k \ldots R_2^{-1} x_2 R_1^{-1} x_1$ is an inconsistent preference cycle of P^s, according to the remaining case* (2 *or* 1).

Proof. It is simple to demonstrate.

Therefore, we shall consider only inconsistent preference cycles in a single sense, i.e., we shall find only the cycles according to case 1, called *positive inconsistent preference cycles*, or according to case 2, called *negative inconsistent preference cycles*. Here, we have chosen the first option.

Theorem. *Let P^s be a strict linguistic preference relation associated to a linguistic preference relation P, then any positive* (*negative*) *inconsistent preference cycle $G = x_1 R_1 x_2 R_2 \ldots x_k R_k x_1$ of $k \geqslant 4$ distinct alternatives imply at least one inconsistent preference cycle of three distinct alternatives.*

Proof. Given in Appendix B.

Therefore, based on the above lemma and theorem we have decided to use only positive inconsistent preference cycles with three distinct alternatives to evaluate our consistency measures, and so, positive inconsistent preference cycle sets with three distinct alternatives of each linguistic preference relation P^k, denoted by C^k, are the output of the detecting process.

4.2. Computing process

In this process, we present two types of linguistic consistency measures, distinguishing between two levels of computation: (i) *level of expert* and, (ii) *level of group of experts*, and between two perspectives of evaluation: (i) *qualitative perspective* and, (ii) *quantitative perspective*:

1. *Individual consistency measures.* Used to evaluate the current consistency degree that an expert e_k presents in his/her opinions. This type of measures help the moderator to advise changes to experts in their opinions during the consensus reaching process. The individual consistency measures that we define are: *the quality-based individual consistency measure* and *the quantity-based individual consistency measure.*

2. *Collective consistency measures.* Used to evaluate the current average consistency degree that group of experts present in their opinions. This type of measures, together with linguistic consensus degrees, help the moderator to decide over the necessity to continue the consensus reaching process. We define two collective consistency measures: *the quality-based collective consistency measure* and *the quantity-based collective consistency measure.*

The consistency measures are obtained as follows.

4.2.1. Individual consistency measures

1. *Quality-based individual consistency measure.* This quality-based measure gives a qualitative perspective of a consistency situation existing in a preference relation provided by one expert. It evaluates the quality of the expert's consistency considered according to the quality of relationships existing between alternatives contained in considered inconsistent cycles, i.e., according to the strict preference value intensities of considered inconsistent cycles. This measure is based on Montero's rationality measure [21, 22]. Montero's rationality measure is based on the acyclicity degree of a preference relation, and it is calculated using numerical weights obtained from the relationships existing between alternatives of all possible consistent cycles of the preference relation. Our measure is based on the non-acyclicity degree and, it is calculated using only linguistic weights obtained from the relationships existing between alternatives of positive inconsistent cycles with three distinct alternatives of relations. Therefore, this measure is a linguistic-weights-based measure, denoted by IC_a^k and obtained as follows: any positive inconsistent preference cycle $c_i^k \in C^k$ presents the following structure $x_t - x_v - x_w - x_t, \forall t, v, w \in \{1, 2, \ldots, n\}$. For each c_i^k, we define its linguistic weight z_i^k as, $z_i^k = \mathrm{Min}\{p_{tv}^s, p_{vw}^s, p_{wt}^s\}$. Thus, IC_a^k is obtained as follows,

$$\mathrm{IC}_a^k = \begin{cases} \mathrm{Neg}(\mathrm{NA}^k) & \text{if } C^k \neq \emptyset, \\ s_T & \text{otherwise,} \end{cases}$$

where $\mathrm{NA}^k = \mathrm{Max}_i\{z_i^k, i = 1, \ldots, \sharp(C^k)\}$ is the non-acyclicity degree of P^k.

2. *Quantity-based individual consistency measure.* This quantity-based measure gives a quantitative perspective of the consistency situation existing in the preference relation provided by one expert. It assesses the quantity of expert's consistencies considered (expressed by the number of consistencies), which, in our case, is done by using positive inconsistent cycles with three distinct alternatives detected in his/her preference relation. Therefore, this measure is an inconsistent-cycles-based measure, denoted by IC_b^k, and obtained using the concept of fuzzy majority, by means of a linguistic quantifier Q^2, as follows: the cardinal of set of all possible cycles with three distinct alternatives in a set of n alternatives c_t is determined by a combination of n elements taken 3 by 3, i.e.,

$$c_t = \binom{n}{3}.$$

Therefore, an expert e_k presents the following rate of inconsistent cycles with three distinct alternatives, $r^k = \sharp(C^k)/c_t$, and then, $\mathrm{IC}_b^k = Q^2(1 - r^k)$. It is clear that these two measures, IC_a^k and IC_b^k, do not have the same sense of assessment of consistency and, even, they may sometimes present a contradictory situation. For example, there may be an expert e_k with only one positive inconsistent cycle with three distinct alternatives with a linguistic weight of s_T, then IC_a^k would be high and, however, IC_b^k would be low. Therefore, we must try to achieve a balance between both measures and, thus, the moderator must use both in the consensus reaching process to advise each expert.

4.2.2. Collective consistency measures

These measures arise intuitively after defining the individual consistency measures. Therefore, there are two of them:

1. *Quality-based collective consistency measure* (CC_a)
2. *Quantity-based collective consistency measure* (CC_b)

They are obtained from individual consistency measures and $\mu_G(k)$, using the LOWA operator and the concept of fuzzy majority symbolized by a linguistic quantifier Q^1, according to the following expressions,

$$CC_a = \phi_{Q^1}((IC_a^1 \wedge \mu_G(1)), \ldots, (IC_a^m \wedge \mu_G(m))),$$

and

$$CC_b = \phi_{Q^1}((IC_b^1 \wedge \mu_G(1)), \ldots, (IC_b^m \wedge \mu_G(m))),$$

respectively.

It is important to note that these measures may be used as a parameter to validate the final solution obtained in the consensus reaching process. Values of collective consistency measures close to s_T indicate a better social rational consensus solution, and values far away from s_T indicate a worse one. In any case, as in the previous section, the moderator must guide the consensus reaching process by considering both collective consistency measures, i.e., achieving a balance between both measures.

This computing process is shown in Fig. 5.

In short, the process described is very useful to the moderator, because its measures allow the moderator to have plentiful information on the current consistency stage. Directly: individual consistency measures provide information about the consistency degree for each expert and its detecting process information about conflict preference values and, the collective consistency measures provide information about global consistency degrees. Indirectly: information about the individuals, who are less consistent in their opinions, and in which preference this occurs, or information about the preferences where inconsistency is high.

Below, we show the use of the consistency measuring process in one step of the consensus formation process, using the example presented in Section 3.

4.3. Application example of consistency measuring process

Assuming the preference relations provided by four experts in Section 3, the respective strict preference relations are:

$$P^{1,s} = \begin{bmatrix} - & \emptyset & IM & \emptyset \\ VLC & - & SC & SC \\ \emptyset & \emptyset & - & IM \\ MC & \emptyset & \emptyset & - \end{bmatrix},$$

$$P^{2,s} = \begin{bmatrix} - & VLC & I & \emptyset \\ \emptyset & - & SC & \emptyset \\ I & \emptyset & - & \emptyset \\ IM & EU & MC & - \end{bmatrix},$$

$$P^{3,s} = \begin{bmatrix} - & SC & EL & \emptyset \\ \emptyset & - & EU & \emptyset \\ \emptyset & \emptyset & - & \emptyset \\ C & IM & IM & - \end{bmatrix},$$

$$P^{4,s} = \begin{bmatrix} - & \emptyset & EU & \emptyset \\ IM & - & \emptyset & VLC \\ \emptyset & VLC & - & \emptyset \\ MC & \emptyset & MC & - \end{bmatrix}.$$

In the detecting process, the detected sets of positive preference intensity cycles with three distinct alternatives are the following:

- Expert 1: $C^1 = \{(x_1, x_3, x_4, x_1)\}$.
- Expert 2: $C^2 = \{(x_1, x_2, x_3, x_1)\}$.
- Expert 3: $C^3 = \{\emptyset\}$.
- Expert 4: $C^4 = \{(x_1, x_3, x_2, x_1), (x_2, x_4, x_3, x_2)\}$.

Then the linguistic consistency measures are:

A. *Individual consistency measures*

A.1. *Quality-based individual consistency measure.* The linguistic weights of the detected cycles are:

- Linguistic weights of C^1: $\{SC\}$.
- Linguistic weights of C^2: $\{I\}$.
- Linguistic weights of C^3: $\{\emptyset\}$.
- Linguistic weights of C^4: $\{EU, VLC\}$,

where, for example, $EU = \mathrm{Min}\{p_{13}^{4s}(EU), p_{32}^{4s}(VLC), p_{21}^{4s}(IM)\}$. Then, the experts' quality-based consistency degrees are:

$$\{IC_a^1 = MC,\ IC_a^2 = C,\ IC_a^3 = C,\ IC_a^4 = EL\}.$$

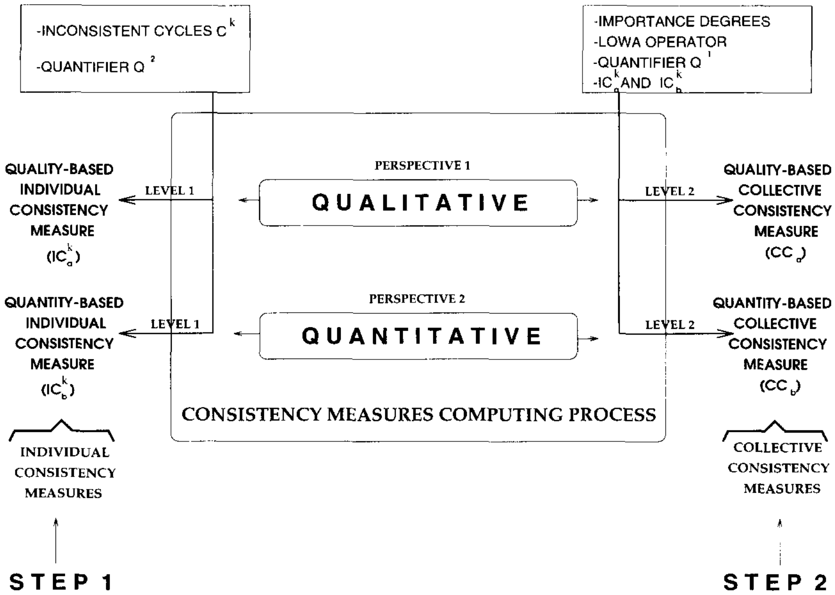

Fig. 5. Consistency computing process.

A.2. *Quantity-based individual consistency measure.* The number of possibles cycles are $c_t = 4$, and thus, the rates of detected cycles are,

- Expert one: $r^1 = 0.25$.
- Expert two: $r^2 = 0.25$.
- Expert three: $r^3 = 0$.
- Expert four: $r^4 = 0.5$,

where, for example, $r^4 = \frac{2}{4}$. Then, the experts' quantity-based consistency degrees using the quantifier Q^2,'at least half', are:

$$\{\mathrm{IC}_b^1 = C,\ \mathrm{IC}_b^2 = C,\ \mathrm{IC}_b^3 = C,\ \mathrm{IC}_b^4 = C\}.$$

B. *Collective consistency measures*

B.1. *Quality-based collective consistency measure.* Using the numerical variant of the linguistic quantifier and the experts' importance degrees, we find:

$$\mathrm{CC}_a = \phi_{Q^1}((MC \wedge EL), (C \wedge C), (C \wedge SC), (EL \wedge EU)) = EL.$$

B.2. *Quantity-based collective consistency measure.* Using the numerical variant of the above linguistic quantifier and importance degrees of experts, we have:

$$\mathrm{CC}_b = \phi_{Q^1}((C \wedge EL), (C \wedge C), (C \wedge SC), (C \wedge EU)) = C.$$

Remarks. According to the concept of fuzzy majority represented by the linguistic quantifier 'at least half' we can draw the following conclusions: (i) from a quantitative view point the expert set is more consistent than from a qualitative view point, (ii) from a quantitative view point all the experts are considered absolutely consistent in their preference values, however, from a qualitative view point only experts e_2 and e_3 are considered in a similar way, (iii) from both the view points the experts e_2 and e_3 are considered absolutely consistent, (iv) if we observe the nature and number of inconsistent preference cycles, expert e_3 is the most consistent one.

5. Conclusions

We have presented a new consensus model in a complete linguistic framework, in group decision making guided by consistency and consensus measures. It includes several linguistic consensus measures and several linguistic consistency measures defined in different action levels. The measures allow analysing, controlling and monitoring the consensus reaching process, describing the current consensus and current consistency stage. Futhermore, consistency measures allow the inconsistencies of the experts' preferences to be detected and the possibility of removing them during the consensus reaching process. So, to sum up, we have defined a new consensus model with more human consistency which is more rational.

Appendix A. Linguistic approach

In this appendix we are going to specify the three essential elements of a linguistic framework considered to develop our rational consensus model for group decision making, i.e., *linguistic preference relations* to express experts' opinions, *the linguistic ordered weighted averaging (LOWA) operator* for aggregating linguistic information used for computing some of our consensus and consistency measures and, *linguistic quantifiers* to represent the concept of fuzzy majority inside of the rational consensus model.

A.1. Linguistic preference relations in group decision making

The use of fuzzy preference relations in decision making situations to express experts' opinions about an alternative set, with respect to certain criteria, appears to be a useful tool in modelling decision processes. Among others, they appear in a very natural way when we want to aggregate experts' preferences into grouped ones, i.e., in the group decision making processes.

As we mentioned earlier, in many cases, an expert is not able to estimate his preference degrees with exact numerical values. So, another possibility is to use linguistic labels, i.e., expressing his opinions about alternatives by means of a *linguistic preference relation*. Therefore, to fix a label set, it is absolutely essential that the experts' preferences be expressed first.

In [2], the use of label sets with odd cardinals was studied, the middle label representing a possibility of 'approximately 0.5', the remaining labels being placed symmetrically around it and the limit of granularity is 11 or no higher than 13. The semantics of the labels is given by fuzzy numbers defined in the $[0,1]$ interval, which are described by membership functions. As the linguistic assessments are merely approximate ones given by the experts, we can consider that linear trapezoidal membership functions are good enough to capture the vagueness of these linguistic assessments, since obtaining more accurate values may be impossible or unnecessary. This representation is achieved by the 4-tuple $(a_i, b_i, \alpha_i, \beta_i)$ (the first two parameters indicate the interval in which the membership value is 1.0; the third and fourth parameters indicate the left and right widths of the distribution).

We shall consider a finite and totally ordered label set $S = \{s_i\}, i \in H = \{0,\dots,T\}$, in the usual sense and with odd cardinality as in [2], where each label s_i represents a possible value for a linguistic real variable, i.e., a vague property or constraint on [0,1]. The following properties are required:

(1) The set is ordered: $s_i \geqslant s_j$ if $i \geqslant$ j.

(2) There is a negation operator: $\mathrm{Neg}(s_i) = s_j$ such that $j = T - i$.

(3) Maximization operator: $\mathrm{Max}(s_i, s_j) = s_i$ if $s_i \geqslant s_j$.

(4) Minimization operator: $\mathrm{Min}(s_i, s_j) = s_i$ if $s_i \leqslant s_j$.

Assuming a linguistic framework and a finite set of alternatives $X = \{x_1, x_2, \dots, x_n\}$, the experts' preference attitude about X can be defined as an nxn linguistic preference relation, such that, $P^k = (p^k_{ij})$, $i,j = 1,\dots,n$, where $p^k_{ij} \in S$ denotes the preference degree of alternative x_i over x_j, linguistically assessed, according to expert's opinion e_k, with

$$s_0 \leqslant p^k_{ij} \leqslant s_T \quad (i,j = 1,\dots,n),$$

and where:

1. $p^k_{ij} = s_T$ indicates the maximum degree of preference of x_i over x_j.
2. $s_{T/2} < p^k_{ij} < s_T$ indicates a definite preference of x_i over x_j.
3. $p^k_{ij} = s_{T/2}$ indicates indifference between x_i and x_j.

A.2. The LOWA operator

An aggregation operator of linguistic information is needed to make good use of the linguistic preference relations for aggregating experts' preferences. Various approaches have been proposed, some use direct computation on labels [7, 9, 29] and, others use computation on associated membership functions [2, 25]. We work following the first approach, which is independent of the semantics of the term set, considering a similar discrimination by the experts. More specifically, we use the linguistic aggregation operator, *linguistic ordered weighted averaging (LOWA)*, defined in [9, 15].

The LOWA operator is based on the *ordered weighted averaging (OWA) operator* defined by Yager [27], and on the *convex combination of linguistic labels* defined by Delgado et al. [7].

Definition. Let $\{a_1, \ldots, a_m\}$ be a set of labels to be aggregated, then the LOWA operator, ϕ, is defined as

$$\begin{aligned}\phi(a_1, \ldots, a_m) &= W \cdot B^{\mathrm{T}} = \mathbb{C}^m\{w_k, b_k, k = 1, \ldots, m\} \\ &= w_1 \odot b_1 \oplus (1 - w_1) \\ &\quad \odot \mathbb{C}^{m-1}\{\beta_h, b_h, h = 2, \ldots, m\},\end{aligned}$$

where $W = [w_1, \ldots, w_m]$, is a weighting vector, such that, $w_i \in [0, 1]$ and $\sum_i w_i = 1$; $\beta_h = w_h / \sum_2^m w_k, h = 2, \ldots, m$, and B is the associated ordered label vector. Each element $b_i \in B$ is the ith largest label in the collection $a_1, \ldots, a_m$. $\mathbb{C}^m$ is the convex combination operator of m labels and if $m = 2$, then it is defined as

$$\begin{aligned}\mathbb{C}^2\{w_i, b_i, i = 1, 2\} &= w_1 \odot s_j \oplus (1 - w_1) \\ &\quad \odot s_i = s_k, s_j, s_i \in S \; (j \geqslant i)\end{aligned}$$

such that

$$k = \min\{T, i + \mathrm{round}(w_1 \cdot (j - i))\},$$

where round is the usual round operation, and $b_1 = s_j$, $b_2 = s_i$.

If $w_j = 1$ and $w_i = 0$ with $i \neq j \; \forall i$, then the convex combination is defined as:

$$\mathbb{C}^m\{w_i, b_i, i = 1, \ldots, m\} = b_j.$$

In [15], we demonstrated that the LOWA operator presents some evidence of rational aggregation, because, on the one hand, it verifies these properties:

- The LOWA operator is *increasing monotonically* with respect to the argument values.
- The LOWA operator is *commutative*.
- The LOWA operator is an '*orand*' operator.

And, on the other hand, it verifies these axioms: *Unrestricted domain, Unanimity or Idempotence, Positive association of social and individual values, Independence of irrelevant alternatives, Citizen sovereignty, Neutrality.*

A.3. How to calculate the weights of the LOWA operator?

A natural question when defining the LOWA operator is, how to obtain the associated weighting vector. In [27, 30], Yager proposed two ways for doing so. The first approach is to use some kind of learning mechanism using sample data; and the second approach is to try to give some semantics or meaning to the weights. The later possibility has allowed multiple applications in the fields of fuzzy and multivalued logics, evidence theory, design of fuzzy controllers, and quantifier guided aggregations. We are interested in the field of quantifier guided aggregations, because our idea is to calculate weights using linguistic quantifiers for representing the concept of *fuzzy majority* in the aggregations that are made in our rational consensus model. Therefore, in the aggregations of the LOWA operator, the concept of fuzzy majority is shown by means of the weights.

In [27, 30], Yager suggested an interesting way to compute the weights of the OWA aggregation operator using linguistic quantifiers, which, in the case of a non-decreasing proportional quantifier Q, is given by this expression:

$$w_i = Q(i/n) - Q((i-1)/n), \quad i = 1, \ldots, n,$$

where the membership function of Q can be represented as

$$Q(r) = \begin{cases} 0 & \text{if } r < a, \\ \dfrac{r-a}{b-a} & \text{if } a \leqslant r \leqslant b, \\ 1 & \text{if } r > b, \end{cases}$$

with $a, b, r \in [0, 1]$. When a fuzzy linguistic quantifier Q is used to compute the weights of the LOWA operator ϕ, it is symbolized by ϕ_Q.

In order to create a more flexible framework, we shall use two types of relative quantifiers. One, with a numerical value described above, and denoted Q^1, and the other one, described in [14], with a linguistical value in a label set $L = \{l_i\}, i \in J = \{0,\dots,U\}$, and denoted Q^2,

$$Q^2 : [0,1] \to L$$

and defined as follows,

$$Q^2(r) = \begin{cases} l_0 & \text{if } r < a, \\ l_i & \text{if } a \leqslant r \leqslant b, \\ l_U & \text{if } r > b, \end{cases}$$

l_0 and l_U are the minimum and maximum labels in L, respectively, and

$$l_i = \mathrm{Sup}_{l_q \in M}\{l_q\},$$

$$\text{with } M = \left\{ l_q \in L : \mu_{l_q}(r) = \mathrm{Sup}_{t \in J}\left\{ \mu_{l_t}\left(\frac{r-a}{b-a}\right)\right\}\right\},$$

with $a,b,r \in [0,1]$. Another definition of Q^2 can be found in [29].

Appendix B. Demonstration of the theorem

We shall demonstrate this only for positive cycles, but it is similar for negative cycles. The demonstration is done by induction about the number of distinct alternatives (k).

- For $k = 4$.

Let $G = x_1R_1x_2R_2x_3R_3x_4R_4x_1$ be a positive inconsistent preference cycle. From the definition of a positive inconsistent preference cycle, we know that at least $\exists h \in \{1,2,3,4\}$, such that $R_h = R$. From the cases:

1. If $h = 1$ then, without loss of generality, we can consider the chain $G' = x_1 - x_2 - x_3 - x_1$, which presents the following structure of relationships, $x_1Rx_2R_2x_3R^?x_1$, with $R_2 \in \{R,I\}$ and $R^? \in \{R,NR,I\}$. Then, from the cases:

(a) If $R^? \in \{R,I\}$ then clearly the considered chain, $G' = x_1 - x_2 - x_3 - x_1$, is a positive inconsistent preference cycle with three distinct alternatives.

(b) If $R^? = NR$ then clearly the chain $G'' = x_1 - x_4 - x_3 - x_1$ is a negative inconsistent preference cycle with three distinct alternatives, which presents the following structure of relationships, $x_1R_4^{-1}x_4R_3^{-1}x_3\ NR\ x_1$.

2. If $h = 2$, then, without loss of generality, we can consider the chain $G' = x_2 - x_3 - x_4 - x_2$, which presents the following structure of relationships, $x_2Rx_3R_3x_4R^?x_2$, with $R_3 \in \{R,I\}$ and $R^? \in \{R,NR,I\}$. Then, from the cases:

(a) If $R^? \in \{R,I\}$ clearly the considered chain $G' = x_2 - x_3 - x_4 - x_2$ is a positive inconsistent preference cycle with three distinct alternatives.

(b) If $R^? = NR$ then clearly the chain $G'' = x_1 - x_4 - x_2 - x_1$ is a negative inconsistent preference cycle with three distinct alternatives, which presents the following structure of relationships, $x_1R_4^{-1}x_4NRx_2R_1^{-1}x_1$.

3. If $h \in \{3,4\}$ then, without loss of generality, we can consider the chain $G' = x_1 - x_3 - x_4 - x_1$, which presents the following structure of relationships, $x_1R^?x_3R_3x_4R_4x_1$, with $R_h = R$, and $R_i \in \{R,I\}$, $i \in \{3,4, i \neq h\}$. Then, from the cases:

(a) If $R^? \in \{R,I\}$ then clearly the considered chain $G' = x_1 - x_3 - x_4 - x_1$ is a positive inconsistent preference cycle with three distinct alternatives.

(b) If $R^? = NR$ then clearly the chain $G'' = x_1 - x_3 - x_2 - x_1$ is a negative inconsistent preference cycle with three distinct alternatives, which presents the following structure of relationships, $x_1NRx_3R_2^{-1}x_2R_2^{-1}x_1$.

Therefore, if there is a positive inconsistent preference cycle with four distinct alternatives, then there is at least one inconsistent preference cycle with three distinct alternatives.

- Suppose that this is true for $k-1$ and, *i.e., if there is a positive inconsistent preference cycle of $k-1$ distinct alternatives then there exist at least one inconsistent preference cycle of three distinct alternatives.*
- For k.

Let $G = x_1R_1x_2R_2\dots R_{k-1}x_kR_kx_1$ be a positive inconsistent preference cycle detected in P^s. Then, by definition of the positive inconsistent preference cycle we know that at least $\exists h \in \{1,2,\dots,k\}$, such that, $R_h = R$. From the cases:

1. If $h = 1$ then, without loss of generality, we can consider the chain $G' = x_1 - x_2 - \cdots - x_{k-1} - x_1$, which presents the following structure of relationships,

$$x_1Rx_2R_2\dots R_{k-2}x_{k-1}R^?x_1,$$

with $R_i \in \{R,I\}$, $i = 2,\dots,k-2$, and $R^? \in \{R,NR,I\}$. Then, from the cases:

(a) If $R^? \in \{R,I\}$ then clearly the considered chain $G' = x_1 - x_2 - \cdots - x_{k-1} - x_1$ is a positive inconsistent preference cycle with $k-1$ distinct alternatives, and by using the induction hypothesis, then G' implies at least one inconsistent preference cycle with three distinct alternatives and thus, G implies at least one inconsistent preference cycle with three distinct alternatives.

(b) If $R^? = NR$ then clearly the chain $G'' = x_1 - x_k - x_{k-1} - x_1$ is a negative inconsistent preference cycle with three distinct alternatives, which presents the following structure of relationships, $x_1 R_k^{-1} x_k R_{k-1}^{-1} x_{k-1}\ NR\ x_1$.

2. If $h = 2$ then, without loss of generality, we can consider the chain $G' = x_2 - x_3 - x_4 - \cdots - x_k - x_2$, which presents the following structure of relationships,

$$x_2 R x_3 R_3 x_4 \cdots R_{k-1} x_k R^? x_2,$$

with $R_i \in \{R, I\}$, $i = 3, \ldots, k-1$, and $R^? \in \{R, NR, I\}$. Then, from the cases:

(a) If $R^? \in \{R, I\}$ then clearly the considered chain $G' = x_2 - x_3 - x_4 - \cdots - x_k - x_2$ is a positive inconsistent preference cycle with $k-1$ distinct alternatives, and by using the induction hypothesis, then G' implies at least one inconsistent preference cycle with three distinct alternatives and thus, G implies at least one inconsistent preference cycle with three distinct alternatives.

(b) If $R^? = NR$ then clearly the chain $G'' = x_1 - x_k - x_2 - x_1$ is a negative inconsistent preference cycle with three distinct alternatives, which presents the following structure of relationships, $x_1 R_k^{-1} x_k\ NR\ x_2 R_1^{-1} x_1$.

3. If $h \in \{3, \ldots, k\}$ then, without loss of generality, we can consider the chain $G' = x_1 - x_3 - x_4 - \cdots - x_k - x_1$, which presents the following structure of relationships,

$$x_1 R^? x_3 R_3 x_4 \ldots R_{k-1} x_k R_k x_1,$$

with $R_h = R$ and $R_i \in \{R, I\}$, $i = 3, \ldots, k, i \neq h$. Then, from the cases:

(a) If $R^? \in \{R, I\}$ then clearly the considered chain $G' = x_1 - x_3 - x_4 - \cdots - x_k - x_1$ is a positive inconsistent preference cycle of $k-1$ distinct alternatives, and by using the induction hypothesis, then G' implies at least one inconsistent preference cycle with three distinct alternatives and thus, G implies at least one inconsistent preference cycle with three distinct alternatives.

(b) If $R^? = NR$ then clearly the chain $G'' = x_1 - x_3 - x_2 - x_1$ is a negative inconsistent preference cycle with three distinct alternatives, which presents the following structure of relationships, $x_1 NR x_3 R_2^{-1} x_2 R_1^{-1} x_1$.

Reference

[1] J.C. Bezdek, B. Spillman and R. Spillman, A fuzzy relation space for group decision theory, *Fuzzy Sets and Systems* **1** (1978) 255–268.

[2] P.P. Bonissone and K.S. Decker, Selecting uncertainty calculi and granularity: an experiment in trading-off precision and complexity, in: L.H. Kanal and J.F. Lemmer, Eds., *Uncertainty in Artificial Intelligence* (North-Holland, Amsterdam, 1986) 217–247.

[3] C. Carlsson, D. Ehrenberg, P. Eklund, M. Fedrizzi, P. Gustafsson, P. Lindholm, G. Merkuryeva, T. Riissanen and A.G.S. Ventre, Consensus in distributed soft environments, *European J. Oper. Res.* **61** (1992) 165–185.

[4] V. Cutello and J. Montero, A model for amalgamation in group decision making, in: J. Villareal, Ed., *NAFIPS'92*, Vol. 1 (NASA Conference Publications, Houston, 1992) 215–223.

[5] V. Cutello and J. Montero, Fuzzy rationality measures, *Fuzzy Sets and Systems* **62** (1994) 39–54.

[6] M. Delgado, J.L. Verdegay and M.A. Vila, Linguistic decision making models, *Internat. J. Intelligent Systems* **7** (1993) 479–492.

[7] M. Delgado, J.L. Verdegay and M.A. Vila, On aggregation operations of linguistic labels, *Internat. J. Intelligent Systems* **8** (1993) 351–370.

[8] M. Delgado, J.L. Verdegay and M.A. Vila, A model for incomplete and vague information in decision making problems, *Internat. J. Intelligent Systems* **9** (1994) 365–378.

[9] F. Herrera and J.L. Verdegay, Linguistic assessments in group decision, in: Proc. First European Congress on Fuzzy and Intelligent Technologies, Aachen (1993) 941–948.

[10] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, A linguistic decision process in group decision making, *Group Decision and Negotiation* **5** (1996) 165–176.

[11] F. Herrera and J.L. Verdegay, On group decision making under linguistic preferences and fuzzy linguistic quantifiers, *Proc. Fifth Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, Paris (1994) 418–422.

[12] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, On dominance degrees in group decision making with linguistic preferences, in: *Proc. Fourth Internat. Workshop, Current Issues on Fuzzy Technologies: Decision Models and Systems*, Trento (1994) 113–117.

[13] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, A sequential selection process in group decision making with linguistic assessment, *Inform. Sci.* **85** (1995) 223–239.

[14] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, A model of consensus in group decision making under linguistic assessments, *Fuzzy Sets and Systems* **78** (1996) 73–87.

[15] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, Direct approach processes in group decision making using linguistic OWA operators, *Fuzzy Sets and Systems* **79** (1996) 175–190.
[16] J. Kacprzyk and M. Fedrizzi, A 'soft' measure of consensus in the setting of partial (fuzzy) preferences, *European J. Oper. Res.* **34** (1988) 316–323.
[17] J. Kacprzyk and M. Roubens, *Non-Conventional Preference Relations in Decision Making* (Springer, Berlin, 1988).
[18] J. Kacprzyk and M. Fedrizzi, *Multiperson Decision Making Models Using Fuzzy Sets and Possibility Theory* (Kluwer Academic Publishers, Dordrecht, 1990).
[19] W.J.M. Kickert, *Fuzzy Theories on Decision Making* (Nijhoff, London, 1978).
[20] L. Mich, L. Gaio and M. Fedrizzi, On fuzzy logic-based consensus in group decision, *Proc. Fifth IFSA World Congress*, Seoul (1993) 698–700.
[21] J. Montero, Arrow's theorem under fuzzy rationality, *Behavioural Sci.* **32** (1987) 267–273.
[22] J. Montero, Social welfare functions in a fuzzy environment, *Kybernetes* **16** (1987) 241–245.
[23] S.A. Orlovski, Decision making with a fuzzy preference relation, *Fuzzy Sets and Systems* **1** (1978) 155–167.
[24] A.K. Sen, Social choice theory: a re-examination, *Econometrica* **45** (1977) 53–89.
[25] M. Tong and P.P. Bonissone, A linguistic approach to decision making with fuzzy sets, *IEEE Trans. Systems Man Cybernet.* **10** (11) (1980) 716–723.
[26] V. Torra and U. Cortes, Towards an automatic consensus generator tool: EGAC, *IEEE Trans. Systems Man and Cybernet.* **25**(5) (1995) 888–894.
[27] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, *IEEE Trans. Systems Man Cybernet.* **18**(1) (1988) 183–190.
[28] R.R. Yager, Fuzzy screening systems, in: R. Lowen, Ed., *Fuzzy Logic: State of the Art* (Kluwer Academic Publishers, Dordrecht, 1993) 251–261.
[29] R.R. Yager, Applications and extension of OWA aggregation, *Internat. J. Man–Machine Studies* **37** (1992) 103–132.
[30] R.R. Yager, Families of OWA operators, *Fuzzy Sets and Systems* **59** (1993) 125–148.
[31] L.A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, *Comput. Math. Appl.* **9** (1983) 149–184.
[32] H.J. Zimmermann, *Fuzzy Sets Theory and its Applications* (Kluwer-Nijhoff, Boston, 1985).