ELSEVIER

Fuzzy Sets and Systems 86 (1997) 15–41

FUZZY sets and systems

Applicability of the fuzzy operators in the design of fuzzy logic controllers

O. Cordón, F. Herrera*, A. Peregrín

Department of Computer Science and Artificial Intelligence, Escuela Técnica Superior de Ingeniería Informática, University of Granada, 18071 Granada, Spain

Received May 1995; revised October 1995

Abstract

A study of the different roles played by the fuzzy operators in fuzzy control is developed in this paper. The behavior of a very large amount of fuzzy operators is analyzed and a comparison of the accuracy of many fuzzy logic controllers designed by means of these operators is carried out. In order to do that, a comparison methodology is defined and two fuzzy control applications are selected, the Inverted Pendulum problem and the fuzzy modeling of the real curve $Y = X$.

Keywords: t-norm; t-conorm; Fuzzy implication operator; Fuzzy inference; Defuzzification; Fuzzy logic controller

1. Introduction

The purpose of any controller is to look periodically at the values of the state variables of the controlled system and to obtain the values associated to their control variables by means of the relationships existing between them. If these relationships can be expressed in a mathematical way, it is not too difficult to design the controller. The problem comes when, as it happens in a lot of real world nonlinear systems with complex dynamics, there is no mathematical model representing the existing relationships.

Fuzzy Logic Control is the main topic of this new field known as *Expert Control*. *Fuzzy Logic Controllers* (FLCs), initiated by Mamdani and Assilian [26], are now considered as one of the most important applications of the *Fuzzy Set Theory* suggested by Zadeh in 1965 [41]. FLCs are knowledge based controllers usually derived from a knowledge acquisition process or automatically synthesized from a self-organizing control architecture [4].

During the past years, many applications of Fuzzy Logic Control have been developed successfully (see [18, 4, 21, 3]) and FLCs have been proved to be superior in performance to conventional systems in many applications.

An FLC is composed by a *Knowledge Base*, that comprises the information given by the process operator in the form of linguistic control rules, a *Fuzzification Interface*, which has the effect of transforming crisp data into fuzzy sets, an *Inference System*, that uses them joined to the Knowledge Base to make inference by means of a reasoning method, and a *Defuzzification Interface*, that

* Corresponding author.

SSDI 0165-0114(95)00367-3

translates the fuzzy control action into a real control action using a defuzzification method. The generic structure of an FLC is shown in Fig. 1.

Complete information about FLCs can be found in [3, 15, 21].

As is known, the accuracy of an FLC depends directly on the following two factors:

(a) *The composition of the FLC Knowledge Base*, that is, the set and type of the fuzzy control rules forming the Rule Base [15, 21] and the scaling factors, the number of linguistic terms in the fuzzy partition of the input and output spaces and the shape of the linguistic variables membership functions of the primary fuzzy sets collected in the Data Base [2, 6, 9, 17].

(b) *The design parameters of the Inference Mechanism.*

In this paper we will focus on the second one, taking as base a generic Knowledge Base constituted by m Mamdani type fuzzy control rules $R_i, i = 1, \dots, m$, with the form:

$$\begin{array}{l} \textit{If } X_{11} \textit{ is } A_{11} \textit{ and } \dots \textit{ and } X_{1n} \textit{ is } A_{1n} \textit{ then } Y \textit{ is } B_1 \\ \quad \textit{also} \\ \quad \cdots \\ \quad \textit{also} \\ \textit{If } X_{m1} \textit{ is } A_{m1} \textit{ and } \dots \textit{ and } X_{mn} \textit{ is } A_{mn} \textit{ then } Y \textit{ is } B_m. \end{array} \tag{1.1}$$

There are several parameters which have a significant influence in the behavior of this FLC component [20]:

1. The form of the mathematical definition of the fuzzy implication in the fuzzy control rules (*If X is A then Y is B*), that is, the selection of the fuzzy implication operator I representing the fuzzy relation R existing between A and B and defined in $\boldsymbol{X} \times \boldsymbol{Y}$ ($\boldsymbol{X}$ and $\boldsymbol{Y}$ being the universes of the variables X and Y, respectively):

$$\forall x \in \boldsymbol{X},\ y \in \boldsymbol{Y}: \quad \mu_R(x, y) = I(\mu_A(x), \mu_B(y)). \tag{1.2}$$

2. The form of the mathematical definition of the sentence connective *and*, that is, the selection of the conjunctive operator T to be used when the fuzzy control rules have more than one variable in the antecedent part (like the ones shown in expression 1.1):

$$\mu_A(x_0) = T(\mu_{A1}(x_1), \dots, \mu_{An}(x_n)), \tag{1.3}$$

where $x_0 = (x_1, \dots, x_n)$.

3. The form of the mathematical definition of composition of fuzzy relations existing in the Compositional Rule of Inference (CRI):

$$\mu_{B'}(y) = \operatorname*{Sup}_{x \in X} \{T'(\mu_{A'}(x), I(\mu_A(x), \mu_B(y)))\}. \tag{1.4}$$

4. The form of the mathematical definition of the sentence connective *also*, that is, the selection of the aggregation operator U (see expression 1.1):

$$\mu_{B'}(y) = U\{\mu_{B'1}(y), \dots, \mu_{B'n}(y)\}. \tag{1.5}$$

5. The way of defining the defuzzification operator D:

$$y_0 = S(x_0) = D(\mu_{B'}(y)). \tag{1.6}$$

The purpose of this paper is to study the behavior of a large amount of fuzzy operators proposed in the specialized literature in the different fuzzy control roles mentioned, complementing the analysis developed until now. In order to put this into effect we design a lot of FLCs combining these operators and define a comparison methodology based on different FLC performance measures that

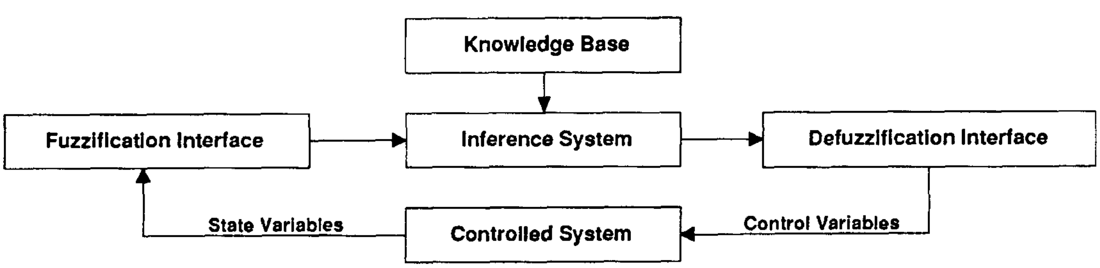

Fig. 1. Generic structure of a fuzzy logic controller.

allow us to determine their accuracy in two different problems, the fuzzy control of the Inverted Pendulum and the fuzzy modeling of the mathematical function $Y = X$. In this way, we consider a difficult nonlinear problem widely studied in Control Theory requiring a quick and accurate FLC response [40] joined to a very simple problem that avoids losing the generality of the fuzzy model and makes the system behavior more dependent on the concrete FLC accuracy [6].

In order to do that, this paper is set up as follows. The next section presents a short review of the studies developed until now based on the use of fuzzy operators in FLCs. Section 3 introduces some preliminaries as the t-norm, t-conorm and implication function definitions. Section 4 analyzes more widely the Defuzzification Interface, presenting two different modes of operation and several defuzzification strategies for each one of them. A comparison methodology of the different FLCs behavior is defined in Section 5, while the fuzzy control applications used for developing our study are presented in Section 6. Section 7 includes the experimental results obtained, which are discussed in Section 8. Finally, several concluding remarks are commented on in Section 9.

2. Previous studies on the use of fuzzy operators in the design of fuzzy logic controllers

Due to the input x corresponding to the state variables of the controlled system is crisp, $x = x_0$, the application of the Fuzzification Interface makes the fuzzy set A' to be a singleton, that is, $\mu_{A'}(x) = 1$ if $x = x_0$ and $\mu_{A'}(x) = 0$ for $x \neq x_0$. Thus the CRI is reduced to the following expression:

$$\mu_{B'}(y) = I(\mu_A(x_0), \mu_B(y)). \tag{2.1}$$

Hence it is found that it depends directly on the implication operator selected. In the specialized literature it is proposed that a huge amount of operators can be used as implication operators in the fuzzy control inference process. Many authors have presented and analyzed several implication operators such as: implications introduced from many-valued logic systems [29], implication functions [34, 24], t-norms [28, 12, 13] and a wide range of other kind of implications [20, 5, 6]. Analyzing these works, it is possible to draw the conclusion that the selection of the best implication operator has become one of the principal problems of inference in fuzzy control.

Many studies adding some information in order to select this operator have been developed in the specialized literature. In [29], Mizumoto and Zimmerman introduced some implication operators of many-valued logic systems and studied their behavior in fuzzy reasoning based on the GMP and on the Generalized Modus Tollens when the inputs to the Inference System are fuzzy concepts. In a later work, Mizumoto [27] analyzed the accuracy of several of these inference operators in the fuzzy control of a plant model. Kiszka and his colleagues [20] collected 36 implication operators and studied their accuracy in the fuzzy modeling of a d.c. series motor. Cao and Kandel [5, 6] defined a new methodology of comparison and analyzed the behavior of the operators employed by Kiszka using them in the fuzzy modeling of different mathematical functions. Finally, in [13], Gupta and Qi studied the behavior of several implication operators based on t-operators in the same problem considered by Mizumoto. Other studies have been carried out in [22, 30, 37, 7, 8, 10].

Several of these works analyze the other five factors discussed. In [20, 5, 6, 22], the mathematical definition of the connective *also* is studied, using the operators Max and Min in this role. In the first two works, 72 different inference processes are composed using these two aggregation operators and the 36 implication operators collected. In [13] different t-operators are used for the same purpose and to represent the connective *and* of the control rules. In [8] six t-norms are used to represent this connective in the fuzzy control of the Inverted Pendulum problem. Finally, in [14–16, 19, 31, 32, 7] a wide variety of defuzzification methods are presented and deeply studied.

3. Preliminaries: t-norms, t-conorms, implication functions and other implication operators

In this section we introduce the definition of the fuzzy operators that have been proposed in the specialized literature to be used in the fuzzy control inference process.

A function $T:[0,1]\times[0,1]\to[0,1]$ is a *t-norm* iff $\forall x,y,z\in[0,1]$ verifies the following properties [28, 12]:

(T1) Existence of a unit 1: $T(1,x)=x$.
(T2) Monotonicity: If $x\leqslant y$ then $T(x,z)\leqslant T(y,z)$.
(T3) Commutativity: $T(x,y)=T(y,x)$.
(T4) Associativity: $T(x,T(y,z))=T(T(x,y),z)$.
(T5) $T(0,x)=0$.

A function $S:[0,1]\times[0,1]\to[0,1]$ is a *t-conorm* iff $\forall x,y,z\in[0,1]$ verifies the following properties [28, 12]:

(S1) Existence of a unit 0: $S(0,x)=x$.
(S2) Monotonicity: If $x\leqslant y$ then $S(x,z)\leqslant S(y,z)$.
(S3) Commutativity: $S(x,y)=S(y,x)$.
(S4) Associativity: $S(x,S(y,z))=S(S(x,y),z)$.
(S5) Existence of a unit 1: $S(1,x)=1$.

T-norms and T-conorms (triangular norms and conorms respectively) were introduced and studied in the context of statistical metric spaces. Later, several authors introduced both operators into Fuzzy Set Theory to represent the intersection and union of fuzzy sets [1].

We employ them in different FLC roles in this paper. Both operators will be used to define the aggregation operator *also* (expression 1.5) as well as the relation R existing in the Knowledge Base control rules as an implication operator (expression 1.2). On the other hand, t-norms will be used to define the connective *and* of expression 1.3 as well.

Regarding the t-norms, in this work we use for the first purpose Zadeh's conventional connective, Minimum. For the other two roles we have selected the following six t-norms: Logical (T1, I8) (Minimum), Hamacher (T2, I40), Algebraic (T3, I25), Einstein (T4, I41), Bounded (T5, I31) and Drastic (T6, I10) Products. Moreover, with regard to the t-conorms, the one used to define the sentence connective *also* will be Maximum and the selected ones that will be used as implication operators are: Logical (I32) (Maximum), Algebraic (I23), Bounded (I30) and Drastic (I33) Sums. The mathematical expressions of all the t-operators selected can be found in Appendix A.

A continuous function $I:[0,1]\times[0,1]\to[0,1]$ is an *implication function* iff $\forall x,x',y,y',z\in[0,1]$ verifies the following properties [34]:

(I1) If $x\leqslant x'$ then $I(x,y)\geqslant I(x',y)$.
(I2) If $y\leqslant y'$ then $I(x,y)\leqslant I(x,y')$.
(I3) Falsity Principle: $I(0,x)=1$.
(I4) Neutrality Principle: $I(1,x)=x$.
(I5) Exchange Principle: $I(x,I(y,z))=I(y,I(x,z))$.

The implication functions have been widely studied by many authors as ways to define the implication relation existing in the GMP rules [34, 24]. They are classified in the different following families:

- *Strong implications or S-implications*: Corresponding to the definition of implication in classical Boolean Logic: $A\to B=\neg A\vee B$. They present the form: $I(x,y)=S(N(a),b)$, S being a t-conorm and N a negation function.
- *Residual implications or R-implications*: Obtained by residuation of a continuous t-norm T in the way $I(x,y)=\mathrm{Sup}\{c\colon c\in[0,1]/T(c,x)\leqslant y\}$.
- *Quantum mechanics implications or QM-implications*: Corresponding to the definition of implication in Quantum Logic: $A\to B=\neg A\vee(A\wedge B)$. Defined in fuzzy logic by means of $I(x,y)=S(N(x),T(x,y))$, S being a t-conorm, N a negation function and T a t-norm.

In this paper we use the implication functions to represent the fuzzy implication in the fuzzy control rules (expression 1.2). For this purpose we selected the following implication functions belonging to the different families above. Their mathematical expressions can be found in Appendix A:

- *S-implications*: Diene (I6), Dubois-Prade (I39) and Mizumoto (I22).
- *R-implications*: Göguen (I4), Gödel (I27) and Lukasiewicz (I5), this last implication function belonging to the S-implications family as well.
- *QM-implications*: Early-Zadeh (I7).

Many operators not belonging to any of these well-defined families have been introduced in fuzzy logic literature to be used as implication operators. We are going to use 24 ones collected in the works [29, 20, 5, 37, 7, 8, 10].

Thus we have 41 operators to be used as implication operators. Thirty-six of them have been collected in [20], so we will name them in the same way used in this work (I1 to I36). The other five implication operators selected will be named from I37 to I41. A study of the mathematical properties satisfied by these operators is presented in Appendix B. In order to put it into effect we analyze the properties verified by the three families of fuzzy operators presented in this section and in Section 2 and collect the different existing ones, denoting them from P1 to P11. O being an implication operator $\forall x, x', y, y', z \in [0, 1]$, they are the following:

(P1) If $x \leqslant x'$ then $O(x,y) \geqslant O(x',y)$.
(P2) If $y \leqslant y'$ then $O(x,y) \leqslant O(x,y')$.
(P3) $O(0,x) = 1$.
(P4) $O(1,x) = x$.
(P5) $O(x,O(y,z)) = O(y,O(x,z))$.
(P6) If $x \leqslant x'$ then $O(x,y) \leqslant O(x',y)$.
(P7) $O(x,y) = O(y,x)$.
(P8) $O(x,O(y,z)) = O(O(x,y),z)$.
(P9) $O(0,x) = 0$.
(P10) $O(0,x) = x$.
(P11) $O(1,x) = 1$.

The last two properties considered in our study, P12 and P13, are obtained from the classification presented in [11] which discriminates the implication operators into two different families, those being an extension of the *boolean implication* (like the implication functions) and those being an extension of the *boolean conjunction* (like the t-norms). Thus an implication operator will verify P12 or P13 if it belongs to any of the two families, that is, if it satisfies one of the following truth tables (clearly, both properties are mutually exclusive):

1. Extensions of the boolean implication (P12):

$a \backslash b$	0	1
0	1	1
1	0	1

2. Extensions of the boolean conjunction (P13):

$a \backslash b$	0	1
0	0	0
1	0	1

4. The Defuzzification Interface

The Defuzzification Interface is the component of the FLC that combines the fuzzy information contained in the individual fuzzy sets inferred and translates it to a crisp control action that will be applied in the controlled system. It is possible to choose between two different modes of operation:

(a) *Mode A: aggregation first, defuzzification after*: In this case the Defuzzification Interface performs the following tasks:

(i) aggregation of the individual fuzzy sets B'_i inferred to get the final output fuzzy set B', by means of the fuzzy operator representing the sentence connective *also*, U.

(ii) defuzzification of this fuzzy set B' which yields a nonfuzzy control action from it, by means of a defuzzification method D.

(b) *Mode B: defuzzification first, aggregation after*: In this second modus operandi the contribution of each fuzzy set inferred is considered individually and the final crisp control action is obtained by taking a calculus (an average, a weighted sum or a selection of one of them) over a concrete crisp Characteristic Value obtained from each one of them. The computation of the final fuzzy set B' is so avoided. This operation constitutes a different approximation to the concept represented by the connective *also*.

With the purpose of developing both modes of operation, we introduce the following terminology [7]. B'_i being the fuzzy set obtained by firing the rule R_i in the inference process and $\mu_{B'i}$ its membership function, we define two different kind of values of significative importance in the Defuzzification process:

1. *The Importance Degrees of the rule* R_i

(a) The *area* of the surface contained by the membership function $\mu_{B'i}$ with the X-axis, s_i:

$$s_i = \int_Y \mu_{B_i'}(y)\,\mathrm{d}y. \tag{4.1}$$

(b) The *height* of B_i', y_i:

$$y_i = \operatorname{Sup} \mu_{B'i}(x), \quad \forall x. \tag{4.2}$$

(c) The matching of the fuzzy sets in the antecedent of the rule R_i, h_i, which depends on the t-norm T used (expression 1.3):

$$h_i = T(\mu_{A1}(x_1), \ldots, \mu_{An}(x_n)). \tag{4.3}$$

2. *The Characteristic Values of the fuzzy set* B_i'

(a) The *maximum value* (MV) of the membership function $\mu_{B'i}$, G_i:

$$G_i = x \in \boldsymbol{X} \mid \mu_{B'i}(x) = y_i. \tag{4.4}$$

When there are more than one x satisfying the condition, it can be obtained in several different ways [14–16]: taking the lower (First-of-Maxima) or the higher (Last-of-Maxima) of them, or taking the average of these two values (Middle-of-Maxima).

(b) The *center of gravity* (CG) of B_i', W_i:

$$W_i = \frac{\int_Y y \cdot \mu_{B_i'}(y)\,\mathrm{d}y}{\int_Y \mu_{B_i'}(y)\,\mathrm{d}y}. \tag{4.5}$$

We obtain the MV by means of the Middle-of-Maxima procedure in order to develop our study.

Taking as base the definitions above and analyzing some works existing in the specialized literature [14–16, 19, 32, 31, 7], several defuzzification methods working in the two different operation modes can be collected.

The defuzzification methods used classically in mode A are those obtaining directly the Characteristic Values of the final aggregated fuzzy set B', that is, the *Center of Gravity* and the *First*, *Last and Middle of Maxima* (*MOM*). In this way, another question that has to be solved in this modus operandi is the selection of the fuzzy operator representing the connective *also*. It is possible to choose between two different options, using a t-norm or a t-conorm in this role. The most used t-operators are the classical Minimum and Maximum operators [20, 5, 6, 22]. In this work we are going to use these two operators for this task and the defuzzification methods Middle of Maxima (D1a) and Center of Gravity (D2a).

Other more complex defuzzification strategies used in mode A can be found in [33, 38, 39].

Regarding the second mode of operation, mode B, most of the defuzzification methods used can be classified into two principal groups of defuzzification methods [7]: *based on the CG and based on the MV*. Both main groups can be subdivided into two subgroups, according to the calculus used to combine the Characteristic Values obtained from every fuzzy set inferred, either using a weighted sum with respect to each concrete Importance Degree or selecting the Characteristic Value of the fuzzy set that presents the largest value of an Importance Degree.

Thus, denoting by y_0 the crisp value obtained from the defuzzification process, we have the following defuzzification methods belonging to each group ($i = 1, \ldots, m$):

- *Weighted Sum with respect to an Importance Degree* [35, 16, 7]:

(D1b) CG weighted by s_i:

$$y_0 = \frac{\sum_i s_i \cdot W_i}{\sum_i s_i}. \tag{4.6}$$

(D2b) CG weighted by y_i:

$$y_0 = \frac{\sum_i y_i \cdot W_i}{\sum_i y_i}. \tag{4.7}$$

(D3b) CG weighted by h_i:

$$y_0 = \frac{\sum_i h_i \cdot W_i}{\sum_i h_i}. \tag{4.8}$$

(D4b) MV weighted by s_i:

$$y_0 = \frac{\sum_i s_i \cdot G_i}{\sum_i s_i}. \tag{4.9}$$

(D5b) MV weighted by y_i:

$$y_0 = \frac{\sum_i y_i \cdot G_i}{\sum_i y_i}. \tag{4.10}$$

(D6b) MV weighted by h_i:

$$y_0 = \frac{\sum_i h_i \cdot G_i}{\sum_i h_i}. \tag{4.11}$$

- *Based on the fuzzy set with largest Importance Degree*:

(D7b) CG of the fuzzy set with largest s_i [14–16, 31, 32, 7]:

$$B'_k = B'_i \,|\, s_i = \mathrm{Max}(s_t),\ \forall t \in \{1, \dots, m\}, \qquad y_0 = W_k. \tag{4.12}$$

(D8b) CG of the fuzzy set with largest y_i [7]:

$$B'_k = B'_i \,|\, y_i = \mathrm{Max}(y_t),\ \forall t \in \{1, \dots, m\}, \qquad y_0 = W_k. \tag{4.13}$$

(D9b) CG of the fuzzy set with largest h_i [7]:

$$B'_k = B'_i \,|\, h_i = \mathrm{Max}(h_t),\ \forall t \in \{1, \dots, m\}, \qquad y_0 = W_k. \tag{4.14}$$

(D10b) MV of the fuzzy set with largest s_i [7]:

$$B'_k = B'_i \,|\, s_i = \mathrm{Max}(s_t),\ \forall t \in \{1, \dots, m\}, \qquad y_0 = G_k. \tag{4.15}$$

(D11b) MV of the fuzzy set with largest y_i [7]:

$$B'_k = B'_i \,|\, y_i = \mathrm{Max}(y_t),\ \forall t \in \{1, \dots, m\}, \qquad y_0 = G_k. \tag{4.16}$$

(D12b) MV of the fuzzy set with largest h_i [7]:

$$B'_k = B'_i \,|\, h_i = \mathrm{Max}(h_t),\ \forall t \in \{1, \dots, m\}, \qquad y_0 = G_k. \tag{4.17}$$

Other defuzzification methods that can be used in the second defuzzification modus operandi are the following:

(D13b) Middle of Maximum (MOM):

$$y_0 = \frac{\sum_i G_i}{m}. \tag{4.18}$$

(D14b) Middle of Greater and Lower MV:

$$G_{\min} = \mathrm{Min}\ G_i, \quad \forall i \in \{1, \dots, m\}, \qquad G_{\max} = \mathrm{Max}\ G_i, \quad \forall i \in \{1, \dots, m\}, \qquad y_0 = \frac{G_{\min} + G_{\max}}{2}. \tag{4.19}$$

(D15b) Center of Sums [14–16]:

$$y_0 = \frac{\int_Y y \cdot \sum_i \mu_{B'_i}(y)\, \mathrm{d}y}{\int_Y \sum_i \mu_{B'_i}(y)\, \mathrm{d}y}. \tag{4.20}$$

To finish this section, we have to point out that using two of the implication operators selected, I3 and I19, the fuzzy set inferred is not continuous. The one inferred by means of I19 presents a membership function with only two nonzero values (concretely, with value 1). In [32], a basic constraint of defuzzification algorithms, a one-element, is presented. This constraint is characterized by a fuzzy set with only a nonzero value. Runkler and Glesner enunciate that the one-element is defuzzified to this single element exactly. In our case, we will work similarly and these fuzzy sets with two nonzero values will be defuzzified to the average of the two elements. Due to the discontinuity that appears using these two implication operators, we will use I19 only in the defuzzification mode B, that is, we will not aggregate fuzzy sets of this kind by means of a t-operator and we will not work with I3. The same problem is presented using I1, I2 and I10 to make inference when the membership functions of the control rules fuzzy sets are triangular-shaped (they make the inferred fuzzy set a crisp value) and it is solved in the same way.

5. Comparison methodology

In order to analyze the behavior of the fuzzy operators selected, different FLCs using them in the different roles discussed are designed. Thus $S[i, j, k, l]$ denotes an FLC using the t-norm T_i representing the connective *and* as conjunctive operator ($i = 1, \dots, 6$) and the implication operator j representing the implication of the fuzzy control rules ($j = 1, \dots, 41$). With regard to the index k, it will be equal to 0 when the FLC employs mode B as defuzzification modus operandi and equal to 1 or 2 when it works in mode A. Its values will be 1 when the Minimum t-norm is used as aggregation operator and 2 when the connective *also* is represented by the t-conorm Maximum. Therefore, $k = 0, 1, 2$. Finally, the index l refers to the defuzzification method used by the FLC. If $k = 0$, then the designed FLC's Defuzzification Interface works in mode A and $l = 1\text{b}, \dots, 15\text{b}$, that is, the 15 mode B defuzzifiers presented in the previous section. Otherwise, if $k \neq 0$ then $l = 1\text{a}, 2\text{a}$, the two mode A ones commented on. On the other hand, the output of the FLC will be denoted by $S[i,j,k,l](x_k)$, x_k being the array of state variables values (inputs) provided by the controlled system.

Our next step is to define the comparison methodology. To develop this purpose, there is need to study some FLC performance measures which allow us to establish several measures of comparison of the accuracy of the FLCs designed. In [7, 15] were defined several performance measures belonging to two families that appear classified in [7] into the two following families:

1. *Measures of Convergence*: These kind of measures are based on the speed of reply of the system. They are employed in regulatory control systems, that is, systems presenting a point of equilibrium (for instance, the inverted pendulum). It is possible to define them through the oscillations produced around this point. In this way, we can define a *Measure of Convergence* (*MC*) as

$$\mathrm{MC}(S[i,j,k,l]) = \frac{\sum_{t_i=m}^{n} |e(t_i)|}{(n-m)/\Delta t}, \tag{5.1}$$

where $e(t_i)$ is the system state at time t_i; $\Delta t = |t_i - t_{i-1}|$ is the amplitude in seconds of the system time unit and m, n are the ends of the interval of time studied.

It is necessary to point out the impossibility of assigning a value to this measure when the FLC $S[i,j,k,l]$ loses the control of the controlled system during the interval of time between m and n. In such a case it is considered that MC takes infinity value.

2. *Measures of Error*: These measures employ a set of system evaluation data formed by N arrays of numerical data, Z_k, constituted by the values of the state variables, x_k, and the corresponding values of the associated control variables, y_k:

$$Z_k = (x_k, y_k), \quad k = 1, \ldots, N. \tag{5.2}$$

In this way, this kind of measures are not limited to be applied only to regulatory control systems. As measures of error, we can consider the following [20, 5–8]:

- Maximum Linear Error (MLE):

$$\mathrm{MLE}(S[i,j,k,l]) = \operatorname*{Max}_{k} |y_k - S[i,j,k,l](x_k)|. \tag{5.3}$$

- Medium Linear Error (LE):

$$\mathrm{LE}(S[i,j,k,l]) = \frac{\sum_{k=1}^{N} |y_k - S[i,j,k,l](x_k)|}{N}. \tag{5.4}$$

- Medium Square Error (SE):

$$\mathrm{SE}(S[i,j,k,l]) = \frac{\frac{1}{2}\sum_{k=1}^{N} (y_k - S[i,j,k,l](x_k))^2}{N}. \tag{5.5}$$

In [15] it was remarked that the choice of a performance measure depends on the type of response that the control system designer wishes to achieve. Taking as base this idea, in [7] were presented several *Measures of Adaptation* obtained as performance indexes associated to the above measures. The purpose of these indexes is making easier the comparison of the accuracy of the different FLCs designed. Thus we define an *Adaptation Degree associated to a performance Measure of Error M* of the FLC $S[i,j,k,l]$, AD_ME$[i,j,k,l]$, by means of the following quotient:

$$\mathrm{Min}\ V = \operatorname*{Min}_{i,j,k,l}(M(S[i,j,k,l])),$$

$$\mathrm{Max}\ V = \operatorname*{Max}_{i,j,k,l}(M(S[i,j,k,l])), \tag{5.6}$$

$$\mathrm{AD\_ME}[i,j,k,l] = 1 - \frac{M(S[i,j,k,l]) - \mathrm{Min}\ V}{\mathrm{Max}\ V - \mathrm{Min}\ V},$$

M being any of the above Measures of Error (MLE, LE, SE).

Regarding the Adaptation Degree associated to a Measure of Convergence, AD_MC$[i,j,k,l]$, it requires a different definition because when any of the analyzed combinations loses the control of the system, the value Max V is equal to infinity and the Adaptation Degree cannot be computed by means of expression 5.6. For this kind of measures, we are going to employ the following definition:

$$\mathrm{Min}\ V = \operatorname*{Min}_{i,j,k,l}(M(S[i,j,k,l])),$$

$$\mathrm{Max\ VMC} = \operatorname*{Max}_{i,j,k,l}(M(S[i,j,k,l])),$$

$$\forall i,j,k,l / M(S[i,j,k,l]) \neq \infty,$$

$$\text{AD\_MC}[i,j,k,l] = 1 - \frac{M(S[i,j,k,l]) - \text{Min } V}{2(\text{Max VMC} - \text{Min } V)}$$

if $M(S[i,j,k,l]) \neq \infty$,

$$\text{AD}[i,j,k,l] = 0$$

if $M(S[i,j,k,l]) = \infty$. (5.7)

Analyzing the above definitions, it is easy to conclude that an FLC will present better behavior when more near to 1 is its value in the respective Adaptation Degree. In this way, the values of both Adaptation Degrees of an FLC $S[i,j,k,l]$ are characterized for being included in the interval $[0,1]$, $\forall i,j,k,l$. The difference among them consists of the fact that, while in the AD_ME the values are equally distributed in the commented interval, the values corresponding to the AD_MC are distributed in the same way in the interval $[0.5,1]$ and only are out of it when the FLC loses the control of the system, in which case the value of the AD_MC associated to the measure is 0. This assumption allows us to distinguish more easily between combinations presenting good and bad behavior in the task of controlling the system.

Another question remarked in [15] is the fact that for more complex control problems, the FLC performance is measured by the level of satisfaction of any number of different goals and constraints. We translate this concept to the field of the Adaptation Degrees defining a performance index based on the combination of different ones associated to several measures.

A particular case is to combine an Adaptation Degree associated to a Measure of Convergence and to a Measure of Error. This new measure has the characteristics of these two discussed families of performance measures. The value of adaptation

Table 1
Definition of Maximum, Minimum and Medium Adaptation Degrees

	Conjunctive operator	Implication operator
Maximum AD	$\text{MAXADC}[i] = \text{Max}_{j,k,l}(\text{AD}[i,j,k,l])$	$\text{MAXADI}[j] = \text{Max}_{i,k,l}(\text{AD}[i,j,k,l])$
Minimum AD	$\text{MINADC}[i] = \text{Min}_{j,k,l}(\text{AD}[i,j,k,l])$	$\text{MINADI}[j] = \text{Min}_{i,k,l}(\text{AD}[i,j,k,l])$
Medium AD	if $k = 0$ then $\text{MEDADC}[i] = \frac{1}{40 \cdot 6} \cdot \sum_{j=1}^{40} \sum_{l=3b}^{15b} \text{AD}[i,j,k,l]$ else $\text{MEDADC}[i] = \frac{1}{40 \cdot 2 \cdot 2} \cdot \sum_{j=1}^{40} \sum_{k=1}^{2} \sum_{l=1a}^{2a} \text{AD}[i,j,k,l]$	if $k = 0$ then $\text{MEDADI}[i] = \frac{1}{5 \cdot 6} \cdot \sum_{i=1}^{5} \sum_{l=3b}^{15b} \text{AD}[i,j,k,l]$ else $\text{MEDADI}[j] = \frac{1}{5 \cdot 2 \cdot 2} \cdot \sum_{i=1}^{5} \sum_{k=1}^{2} \sum_{l=1a}^{2a} \text{AD}[i,j,k,l]$
	Aggregation operator	**Defuzzification method**
Maximum AD	$\text{MAXADA}[k] = \text{Max}_{i,j,l}(\text{AD}[i,j,k,l])$	$\text{MAXADD}[l] = \text{Max}_{i,j,k}(\text{AD}[i,j,k,l])$
Minimum AD	$\text{MINADA}[k] = \text{Min}_{i,j,l}(\text{AD}[i,j,k,l])$	$\text{MINADD}[l] = \text{Min}_{i,j,k}(\text{AD}[i,j,k,l])$
Medium AD	if $k = 0$ then $\text{MEDADA}[k] = \frac{1}{5 \cdot 40 \cdot 6} \cdot \sum_{i=1}^{5} \sum_{j=1}^{40} \sum_{l=3b}^{15b} \text{AD}[i,j,k,l]$ else $\text{MEDADA}[k] = \frac{1}{5 \cdot 40 \cdot 2} \cdot \sum_{i=1}^{5} \sum_{j=1}^{40} \sum_{l=1a}^{2a} \text{AD}[i,j,k,l]$	if $k = 0$ then $\text{MEDADD}[l] = \frac{1}{5 \cdot 40} \cdot \sum_{i=1}^{5} \sum_{j=1}^{40} \text{AD}[i,j,k,l]$ else $\text{MEDADD}[k] = \frac{1}{5 \cdot 40 \cdot 2} \cdot \sum_{i=1}^{5} \sum_{j=1}^{40} \sum_{k=1}^{2} \text{AD}[i,j,k,l]$

that receives a concrete controller $S[i,j,k,l]$ will be more suitable than that which would receive from an only measure of one of the two groups. A performance index of this type was presented in [7] combining the above presented Measure of Convergence and Medium Square Error by means of an average function, obtaining then a *Conjunctive Adaptation Degree* (*CAD*), defined in the following way:

$$\mathrm{CAD}[i,j,k,l] = f(\mathrm{AD\_SE}[i,j,k,l], \mathrm{AD\_MC}[i,j,k,l]), \qquad (5.8)$$

AD_SE and AD_MC being the ADs associated to the measures SE and MC respectively and f satisfying $\mathrm{Min}(x,y) \leqslant f(x,y) \leqslant \mathrm{Max}(x,y)$. The function f selected in that work was

$$f(x,y) = \begin{cases} \dfrac{(x+y)}{2} & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (5.9)$$

Finally, for comparing easily the accuracy of some designed FLC, we use the definitions of the Maximum, Minimum and Medium Adaptation Degree for a conjunctive, implication or aggregation operator and defuzzification method of any of the studied measures. Note that AD can be equal to AD_MC, AD_MLE, AD_LE, AD_SE or any CAD in the definitions included in Table 1. The values m, n, o and p represent the number of conjunctive, implication and aggregation operators and defuzzification methods used to develop the different FLCs (in this work, $i = 1, \ldots, 5$; $j = 1, \ldots, 40$; $k = 0, \ldots, 2$; if $k = 0$ then $l = \{3b, 6b, 9b, 12b, 13b, 15b\}$ and $l = \{1a, 2a\}$ otherwise, as we will see in Section 7).

6. Experiments selected: the Inverted Pendulum and the fuzzy modeling of a mathematical function

Two applications have been selected to study the accuracy of the different FLCs designed using the fuzzy operators presented. On the one hand, the *Inverted Pendulum*, a problem widely studied in Control Theory [40]. On the other hand, we will work in the way proposed by the authors of the papers [5, 6] making a fuzzy modeling of the function $Y = X$.

The Inverted Pendulum system [40] is shown in Fig. 2. On the assumption $|\Theta| \leqslant 30°$, the behavior of the pendulum is managed by the following equation:

$$m\frac{l^2}{3}\omega = \tfrac{1}{2}(-F + mg\sin\Theta - k\omega), \qquad (6.1)$$

where m is the mass of the pendulum, $2l$ its length and $k\omega$ is an approximation of the friction strength.

The system state variables are the pendulum angle, Θ, and the change of angle, ω, whereas the control variable is the force F to apply over its gravity center. The universes of discourse of these variables are the following:

$$\omega \in [-0.8645, 0.8645] \text{ rad/s}, \qquad (6.2)$$

$$\Theta \in [-0.5283, 0.5283] \text{ rad}, \qquad (6.3)$$

$$F \in [-3003.8, 3003.8] \text{ Nw}. \qquad (6.4)$$

In order to develop our study, we have worked with a simulation model of the system using the parameters $m = 5$ kg and $2l = 5$ m.

The linguistic variables are partitioned by using the seven linguistic labels contained in the following set [40, 23]:

$$\{\mathrm{NB}, \mathrm{NM}, \mathrm{NS}, \mathrm{ZR}, \mathrm{PS}, \mathrm{PM}, \mathrm{PL}\}, \qquad (6.5)$$

where N is negative, P is positive, B is big, M is medium, S is small and ZR is zero.

The membership functions corresponding to each element in the linguistic set have been obtained following the methodology proposed in

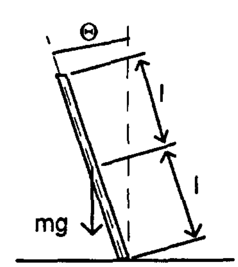

Fig. 2. The Inverted Pendulum.

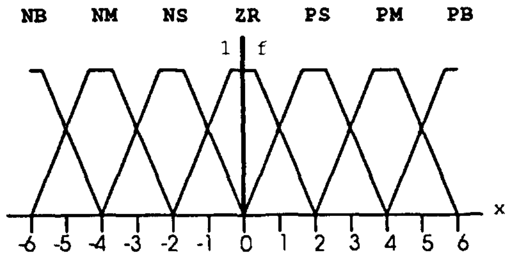

Fig. 3. The domain partition in the Inverted Pendulum problem.

Table 2
Control rule map of F

	Θ						
ω	NB	NM	NS	ZR	PS	PM	PB
NB							
NM							
NS			NS		ZR		
ZR		NM		ZR		PM	
PS			ZR		PS		
PM							
PB							

[23]. The trapezoidal-shaped membership functions shown in Fig. 3 are used scaling the interval $[-6, 6]$ to the one corresponding to the concrete variable (expressions 6.2, 6.3 and 6.4).

The Knowledge Base used to control the system is constituted by the seven linguistic control rules shown in Table 2 [40].

The selection of the second application is based on the studies developed in [6]. The authors enunciate that the independence between the application considered and the accuracy obtained by the FLC is a very important fact in the comparison of the influence of the fuzzy operators used to design it. Hence in order to avoid losing the generality of a fuzzy model, we are going to work with the simplest functional relation $Y = X$, making a fuzzy modeling of this real curve in the interval $[0, 10]$.

In this case, the five following linguistic labels are used to partition the domain of the linguistic variables X and Y:

$$\{\mathrm{VS}, \mathrm{S}, \mathrm{M}, \mathrm{L}, \mathrm{VL}\}, \tag{6.6}$$

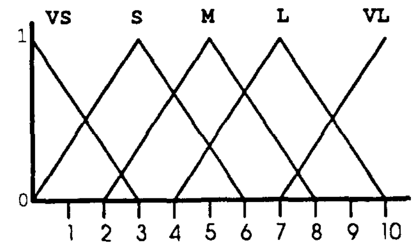

Fig. 4. The domain partition in the fuzzy modeling of function $Y = X$.

where V is very, S is small, M is medium and L is large. The corresponding membership functions, triangular-shaped like in [6], are shown in Fig. 4.

The Knowledge Base presents the following five control rules:

$$\begin{array}{l} \text{If } X \text{ is VS then } Y \text{ is VS} \\ \text{also} \\ \text{If } X \text{ is S then } Y \text{ is S} \\ \text{also} \\ \text{If } X \text{ is M then } Y \text{ is M} \\ \text{also} \\ \text{If } X \text{ is L then } Y \text{ is L} \\ \text{also} \\ \text{If } X \text{ is VL then } Y \text{ is VL.} \end{array} \tag{6.7}$$

7. Experiments and results

The first application, Inverted Pendulum, is characterized to be a regulatory control problem. In this way, it is possible to employ a Measure of Convergence to study the behavior of the FLC used to control it. As we have remarked in Section 5, a performance index combining measures belonging to the two introduced families is a very good measure to determine the accuracy of the different FLCs. We use the Conjunctive Adaptation Degree defined by means of expressions 5.8 and 5.9 to compare the behavior of the FLCs designed in this problem.

On the other hand, the second application does not present a point of equilibrium, so it is not possible to use performance measures belonging to the first family. In this case we will only employ a Measure of Error, the SE, to determine the accuracy of the FLCs.

We will show in this section several tables containing the medium results obtained by the conjunctive, implication and aggregation operators and the defuzzification methods considered in their application to both problems, as well. In order to achieve, we define another Conjunctive Adaptation Degree, CAD_SE, that combines the values of the previously defined CAD in the Inverted Pendulum problem and the AD_SE in the fuzzy modeling of the function $Y = X$ with the same average function used in expression 5.9. We present the values of the Medium CAD_SE for the different operators obtained by means of the expressions defined in Table 1.

The set of evaluation data used to compute the SE have 200 data arrays in the form (value of Θ, value of ω, value of F) belonging to the intervals $\Theta \in [-0.2569, 0.2569]$ rad, $\omega \in [-0.4244, 0.4244]$ rad/s and $F \in [-1474.05, 1474.05]$ Nw in the Inverted Pendulum problem. In the other application selected, this set is constituted by 41 data arrays (value of X, value of Y) with a frequency of 0.25 in the interval $[0, 10]$.

Regarding the fuzzy operators and defuzzification methods presented up to now, we have to point out that not all of them are going to be used in the design of the FLCs studied in this work because it will make this paper excessively extensive. As we have remarked at the end of Section 4, we do not use the implication operators inferring a discontinuous fuzzy set in FLCs whose Defuzzification Interface works in mode A. In the different tables of results shown, these combinations will present an asterisk in the respective cell. Furthermore, it is clear that the t-norms selected to be used as rule antecedent conjunctive operators (expression 1.3) are not employed in the FLCs designed for the second application because the rules antecedent presents only one variable.

As our principal purpose is to study the applicability of fuzzy operators in the different FLC roles, we are going to use only six of the 15 presented mode B defuzzification methods in order not to overload the paper. We take as base the study developed in [7] to make the selection of the six ones. In this paper it is drawn that matching is the most important characteristic of defuzzification at the sight of the average results shown. We select the four defuzzification methods using this Importance Degree (D3b, D6b, D9b and D12b) together with the widely used MOM and Center of Sums (D13b and D15b respectively). On the other hand, in [8] it is pointed out that the sixth t-norm selected, Drastic Product (T6), does not work well when it is used as conjunctive operator in an FLC. As remarked in that work, its bad behavior is probably due to it not being a continuous function, a fact that seems to be a drawback for the fuzzy control inference process, based on interpolative reasoning. For this reason, this t-norm is not used in our study.

It is important to note that, with the purpose of making the paper more readable, we will maintain only the tables containing global results in this section, moving those corresponding to the individual performance measures and indexes (Tables 8–11) to Appendix C. These tables are useful because they show the accuracy of each concrete combination in the application and they are commented on in the following. Note that all tables present the maximum and minimum values remarked in bold, and italics and bold respectively.

Table 8: It includes the Adaptation Degree AD_SE values obtained by the different FLCs in the fuzzy modeling of the function $Y = X$ problem. The minimum and maximum values of the Medium Square Error in all combinations are 0 and 11.9726 respectively, so it is easy to obtain the concrete value of this performance measure for any of the FLCs by means of expression 5.6.

Table 9: The results shown in this table correspond to the Adaptation Degree of the average Medium Square Error obtained by the five FLCs using the t-norms from T1 to T5 as conjunctive operator. In this way, every cell of this table shows the Adaptation Degree associated to the average of the values obtained by the five FLCs using the implication and aggregation operators and defuzzification method determined by its indexes, and the conjunctive operators from T1 to T5, in the commented measure. The minimum and maximum values of the performance measure in the table are 8037.8420 and 351631.0620.

Table 10: The results collected in this table correspond to the Adaptation Degree of the average Measure of Convergence obtained by the five FLCs using the t-norms from T1 to T5 as conjunctive operator in a similar way to that used in the previous table. The main difference is we are not working with the average of the five results now but we are using the following average function:

$$MC[j,k,l] = \frac{\sum_{i=1}^{5} MC[i,j,k,l]}{5}, \quad \text{if } MC[i,j,k,l] \neq \infty, \quad \forall i \qquad (7.1)$$

$$MC[j,k,l] = \infty, \quad \text{otherwise.}$$

Its minimum and maximum values are respectively 51.7399 and 86.9292 and the parameters used to compute the data corresponding to the Measure of Convergence of the different combinations (see the parameters defined in expression 5.1) are the following:

- The limits of the time interval are: $m = 0$ s, $n = 10$ s.
- The extent of the system time unit is: $\Delta t =$ 100 ms.
- The FLC acts every 600 ms.
- The system initial state parameters are: $\Theta =$ 0.25 rad, $\omega = 0.4$ rad/s.

As can be observed, we have selected a very extreme initial system state in order to obtain data that allow us to distinguish between good and bad combinations.

Table 11: This table presents the results corresponding to the Conjunctive Adaptation Degree (CAD) selected for the Inverted Pendulum problem. As noted, it combines the Adaptation Degree associated to the Medium Square Error (AD_SE, Table 9) and to the Measure of Convergence (AD_MC, Table 10) by means of the average function presented in expression 5.9. These results enable us to analyze in the correct way the behavior of the FLCs in the considered problem as they correspond to a performance index containing information about the two families of performance measures commented on. As can be viewed, many of the combinations take value 0 in the defined Conjunctive Adaptation Degree due to their bad results in the Measure of Convergence. This fact allow us to distinguish clearly between good and bad combinations.

The following tables show the medium results for each one of the fuzzy control roles considered in the study. All of them contain average results obtained combining the individual ones corresponding to the two applications selected (Tables 8 and 11) except those collected in Table 3. It is because the conjunctive operators are not used in the fuzzy modeling of the $Y = X$ function problem but, in this case, values corresponding to the two selected measures for the Inverted Pendulum problem are presented. We must point out that the combinations whose values cannot be computed (those presenting a * in their respective cell) are not considered to work out the different medium results. The global results collected in these tables make them adequate to draw several conclusions about the applicability of the fuzzy operators in the different FLC roles, the purpose of this paper.

Hence Table 3 contains the results corresponding to the Medium AD_SE and AD_MC in the Inverted Pendulum problem joined to the average of both indexes for each one of the conjunctive operators selected. The results presented in it have been obtained by means of two tables (one corresponding to the AD_SE and the other to the AD_MC) not included in this paper due to their extension (they both are fourth-dimensional tables $T[i,j,k,l]$ with the indexes i, j, k and l taking values in the intervals shown at the end of Section 5).

Table 4 shows the values of the Medium Adaptation Degrees for the different implication operators. The first column shows the values of the Medium AD_SE for an implication operator in the fuzzy modeling of the function $Y = X$ problem while the second one presents those corresponding to the Medium Conjunctive Adaptation Degree for an implication operator in the Inverted Pendulum problem. The third column presents the average of the data in the two front columns.

Table 3
Medium Adaptation Degrees for a conjunctive operator

	AD_SE	AD_MC	Average
T1	0.75994	***0.21701***	***0.48848***
T2	0.76046	0.24371	0.50208
T3	0.75960	0.22046	0.49003
T4	***0.75707***	**0.25651**	**0.50679**
T5	**0.76245**	0.22516	0.49380

Table 4
Medium Adaptation Degrees for an implication operator

	AD_SE_{y-x}	CAD	Average
I1	0.67781	***0.00000***	0.33890
I2	0.91267	0.31124	0.61196
I4	0.87488	0.41024	0.64256
I5	0.83316	0.15562	0.49439
I6	0.81790	0.23280	0.52535
I7	0.79934	***0.00000***	0.39967
I8	0.98296	**0.81830**	**0.90063**
I9	0.60877	***0.00000***	0.30439
I10	**0.98604**	0.62931	0.80767
I11	0.47924	***0.00000***	0.23962
I12	0.59574	***0.00000***	0.29787
I13	0.45576	***0.00000***	0.22788
I14	0.64784	***0.00000***	0.32392
I15	0.88163	0.39898	0.64031
I16	0.92145	0.08723	0.50434
I17	0.87972	0.48542	0.68257
I18	0.80627	0.31124	0.55876
I19	0.94248	***0.00000***	0.47124
I20	0.58267	***0.00000***	0.29133
I21	0.59921	***0.00000***	0.29960
I22	0.82797	0.25540	0.54169
I23	0.57838	***0.00000***	0.28919
I24	***0.36812***	***0.00000***	***0.18406***
I25	0.97919	0.63666	0.80793
I26	0.81680	***0.00000***	0.40840
I27	0.87405	0.40808	0.64106
I28	0.79595	***0.00000***	0.39797
I29	0.87489	0.39696	0.63593
I30	0.58329	***0.00000***	0.29164
I31	0.92765	0.64322	0.78544
I32	0.56987	***0.00000***	0.28493
I33	0.58252	***0.00000***	0.29126
I34	0.54532	***0.00000***	0.27266
I35	0.59787	***0.00000***	0.29893
I36	0.85922	***0.00000***	0.42961
I37	0.42002	***0.00000***	0.21001
I38	0.79810	***0.00000***	0.39905
I39	0.83590	0.24287	0.53939
I40	0.97866	0.55732	0.76799
I41	0.88124	0.55588	0.71856

Tables 5 and 6 contain the values of the same Medium Adaptation Degrees for the other two operators considered, the defuzzification method and the aggregation operator.

Finally, Table 7 presents several average results corresponding to the implications belonging to the different groups previously commented on in Section 3. Thus, the first row contains the grouped averaged results of all implication operators selected in this work being an extension of the boolean implication. The second one does the same with those belonging to the second family, which includes implication operators being an extension of the boolean conjunction. The last row presents the averaged results obtained by the operators not belonging to any of the two groups.

Table 5
Medium Adaptation Degrees for a defuzzification method

	AD_SE_{y-x}	CAD	Average
D1a	0.65914	0.13458	0.39686
D2a	***0.64459***	***0.11246***	***0.37853***
D3b	0.79930	0.26781	0.53355
D6b	**0.88861**	**0.35022**	**0.61941**
D9b	0.78106	0.23795	0.50951
D12b	0.86406	0.31117	0.58761
D13b	0.74510	0.12889	0.43699
D15b	0.69145	0.10640	0.39892

Table 6
Medium Adaptation Degrees for an aggregation operator

	AD_SE_{y-x}	CAD	Average
$k = 0$	**0.79493**	**0.23374**	**0.51433**
$k = 1$	0.66125	0.13676	0.39901
$k = 2$	***0.64248***	***0.11027***	***0.37638***

8. Analysis of results

In this section we analyze widely the results obtained in the experiments developed. Those corresponding to both experiments separately are studied first, showing the combinations presenting good accuracy and the best operators in each role at the sight to the medium results. The analysis is finished drawing several general conclusions based on the global results averaging those obtained in both applications.

According to the results obtained in the first experiment, the fuzzy modeling of the function

Table 7
Medium Adaptation Degrees for the different families of Implication operators

		AD_SE_{y-x}	CAD	Average	Total
Boolean implications	S-implications	0.82726	0.24369	0.53547	
	R-implications	0.86070	0.32465	0.59267	0.49486
	QM-implications	0.79934	***0.00000***	0.39967	
	Others (I2, I20)	0.74767	0.15562	0.45164	
Boolean conjunctions	T-norms	**0.95596**	**0.64012**	**0.79804**	**0.79804**
Other implications		***0.67545***	0.06719	***0.37132***	***0.37132***

$Y = X$, collected in Table 8, it can be remarked the following:

1. Four combinations using the Minimum (T1) as aggregation operator, the Middle of Maxima (D1a) as defuzzification method and the implication operators I4, I5, I15 and I27 present the best behavior approximating exactly the function $Y = X$, as can be viewed in Fig. 5. Three of these four implication operators (I4, I5 and I27) belong to the family of R-implications, constituted in this way as a good family of implication operators for this application. It is quite strange that several previous studies (such as [5, 6, 20]) show that combining the R-implications I4 and I27 or any of the t-norms with this aggregation operator worse behavior is obtained than when using the Maximum in this last role. In the same way, the good accuracy of the other best implication, I15, is a little strange due to this operator only verifying one of the 13 properties analyzed in this work, as can be viewed in Appendix B.

2. At the sight of the medium values (Table 4, column 1), the implication operators results are quite different. The best implication operator has been the Drastic Product (I10) with little difference with regard to the Minimum (I8), the Algebraic Product (I25) and the Hamacher Product (I40). Thus, the t-norms get better behavior than the R-implications according to the medium results.

3. The defuzzification method working in mode B go clearly beyond those used in mode A, as can be viewed analyzing the Medium ADs for an aggregation operator and for a defuzzification method (Tables 5 and 6). The two methods including the MV and the matching (the MV weighted by the matching (D6b) and the MV of the fuzzy set with largest matching (D12b)) get the best accuracy with a significative difference. It is clear that modus operandi A gives worse results than mode B (both methods working in mode A present the worst behavior) and there is not too much difference between the two aggregation operators selected for working in this mode, value 0.66125 for the Minimum and 0.64248 for the Maximum (Table 6).

The other application selected, the Inverted Pendulum, gives us some new information because it allows us to study the influence of a new operator in fuzzy control, the conjunctive operator, and to use measures belonging to the two families presented to study the designed FLCs behavior, as can be viewed in the following.

It has to be pointed out that the initial system state selected to compute the AD_MC (data collected in Table 10) was very extreme in order to distinguish more clearly between combinations with good and bad accuracy. This implies that FLCs which were working well in previous measures lose the control of the system and take value 0 in the Measure of Convergence. Moreover, it must be considered that the data shown in Table 10 are obtained by means of the average function presented in expression 7.1. Therefore, as long as one of the five FLCs considered (varying the t-norm used as conjunctive operator) loses the control of the system, the result of the AD_MC will be equal to 0.

The following facts can be underlined at the sight of the Inverted Pendulum data obtained:

1. The results corresponding to the AD_SE (Table 9) show that in this case there are five combinations getting the best behavior. Three of them are the same in both applications, the FLCs I4-K1-D1a, I5-K1-D1a and I27-K1-D1a (being used in combination with any of the five t-norms selected). In this way, the R-implications are stronger again than the other implications when they are used in combination with the aggregation operator T1 and the defuzzification method D1a. On the other hand, the FLCs I2-K1-D1a and I2-K2-D2a obtain the best performance as well. They both employ the implication operator I2 whose accuracy was stood up in [20].

2. According to the AD_MC, the best behavior is now presented by the FLCs using Mizumoto's S-implication (I22) as implication operator (underlined too in [5]), the Minimum t-norm as aggregation operator and the Middle of Maxima (D1a) as defuzzification method joined to the five t-norms selected (T1, ..., T5) as conjunctive operators, although the other combinations using this implication operator have very bad accuracy, losing the control of the system in seven of the nine cases. The combinations using a t-norm as implication operator turn out to be very efficient, losing the control of the system only when combined with the aggregation operator Minimum in most of the cases (it is proved in many studies, as [20], that the t-norms do not work well with this aggregation operator when used to make inference in fuzzy control).

3. Analyzing the Medium AD_SE and AD_MC values for a connective (Table 3), we can remark that there is no significative difference among the FLCs designed using the selected t-norms (T1, ..., T5) as conjunctive operators. In the first case, the best behavior corresponds to the t-norm T5, Bounded Product, while in the second one, the best one is the Einstein Product (T4), which curiously presented the worst behavior in the previous measure. With regard to the average results, the most classically used connective, the Minimum (T1), results to be the connective with worst accuracy and the Einstein Product results to be the best one although the differences continue being tiny.

4. The results corresponding to the selected Conjunctive Adaptation Degree (Table 11) are strongly dependent on the AD_MC due to the definition of the average function used (expression 5.9). In this way, the best combination is again I22-K1-D1a, which presents very good behavior in the AD_SE and AD_MC. Combinations using I4 and I27 as implication operators continue giving good behavior in combination with other concrete operators in the other roles, but again the t-norms give the best results, outstanding the fact that none of the FLCs using the Minimum as implication operator get value 0 in the CAD.

From the observation of both the applications average results, the following can be highlighted:

1. The implication operator presenting better behavior (see Table 4, column 3) is the Minimum t-norm I8, with considerable difference with respect to the second, the Algebraic Product, I25 (0.90063 versus 0.80793). There are no changes with the defuzzification methods (Table 5, column 3). The MV weighted by the matching is again the best adapted and the two working in mode A present the worst results (D1a and D2a). Thus, this provokes the mode A, aggregation operators Maximum and Minimum, $K = 1$ and $K = 2$ respectively, to present worse behavior than mode B, with a considerable difference, as can be viewed in the data presented in column 3 of Table 6.

2. Analyzing the results shown in Table 7, several important remarks can be made. We can observe that the implication operators being an extension of the boolean conjunction, that is, in our case, the t-norms, give better accuracy than those belonging to the other family. Regarding the other family of implication operators, the R-implications present the best behavior and the QM-implications the worst (we found similar results in our previous study developed in [7]). The other 25 implications studied do not seem to be useful in fuzzy control according to the averaged results although some concrete FLCs using them can give good accuracy.

To illustrate more clearly the above comments, we present the following three figures showing graphically the behavior of several FLCs with an outstanding accuracy. In Fig. 5, six combinations are compared in the first application. Four of them

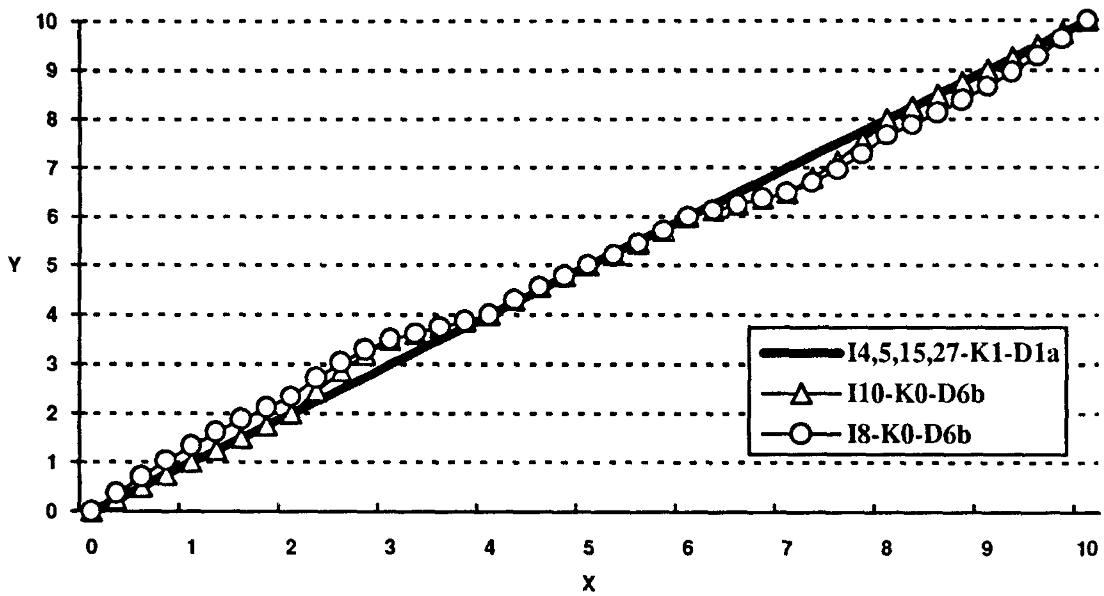

Fig. 5. Graphical representation of the behavior of several FLCs in the fuzzy modeling of the function $Y = X$ problem.

are the ones that have better behavior in this experiment, the FLCs approximating the function $Y = X$ with no error. The fifth is designed selecting the operators presenting better behavior according to the application medium results, the implication operator I10 and the defuzzification method D6b. The last combination is obtained working in a similar way, but selecting the operators with better behavior in the global results, that is, the same defuzzification method and the Minimum t-norm (I8) playing the role of implication operator.

Fig. 6 is based on the same application. In this case, the FLCs studied are designed by means of the best implication operator of every one of the families and subfamilies commented on. As can be viewed studying Appendix B, many of the implication operators selected in this work do not belong to any of the two families of implication operators commented on in Section 3. If we do not take into account the implication functions and the t-norms, only two of the other 27 implications belong to one of the defined families, I2 and I20, implications extending the boolean implication. In this way, the six combinations compared in this figure use as implication operator the following ones: the best of every one of the subfamilies included in this family (S-implications, I22, R-implications, I4, QM-implications, I7, and the group constituted by the other two implications, I2), the best of the ones belonging to the family of the boolean conjunction extensions (that is, the best t-norm at the sight of the global results, I8) and the best of the other implication operators not belonging to any of the two families, I17. They are all combined with the best global defuzzification operator, D6b.

Finally, in Fig. 7 we compare several of the combinations distinguished in our study with the FLC used by Mamdani, T1-I8-K2-D2a (that is, the Minimum as implication and conjunctive operator, the Maximum as aggregation operator and the Center of Gravity as defuzzifier). The other combinations of the figure are selected in a similar way to those which are presented in Fig. 5, all employing the same t-norm used by the Mamdani controller. The best combinations according to the results of the AD_SE, I2,4,5,27-K1-D1a; the best at the sight of the AD_MC and CAD results, I22-K1-D1a; and the combination designed by means of the operators with better behavior, I8-K0-D6b.

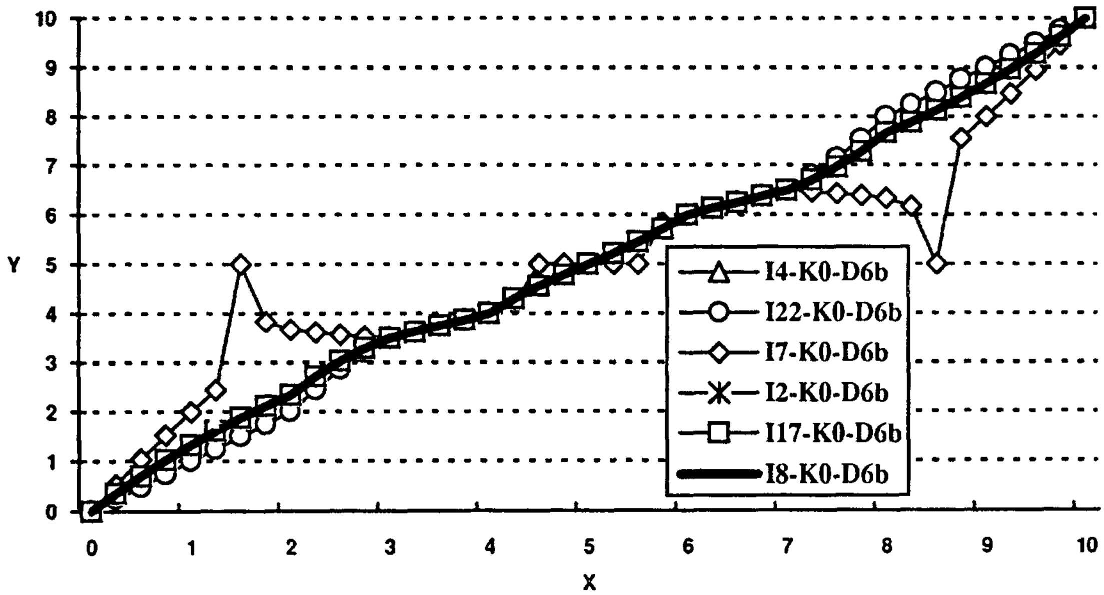

Fig. 6. Graphical representation of the behavior of the FLCs using the best operators according to the average results in the fuzzy modelling of the function $Y = X$ problem.

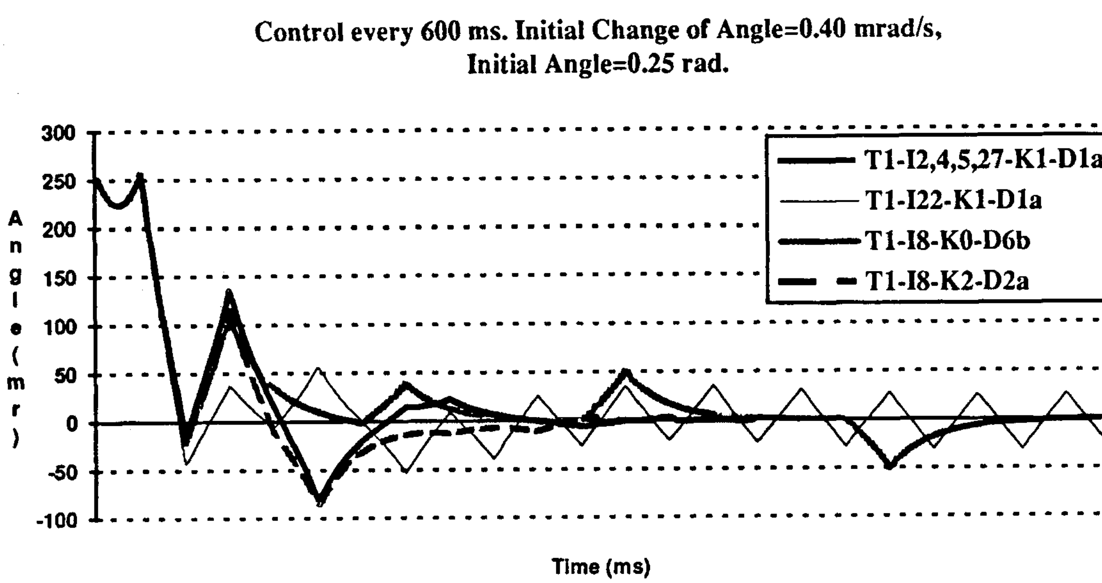

Fig. 7. Graphical representation of the behavior of several FLCs in the Inverted Pendulum problem.

9 Concluding remarks

A study of the different roles played by the fuzzy operators in fuzzy control has been carried out taking as base the problems of the fuzzy modeling of the function $Y = X$ and the Inverted Pendulum. According to the results obtained in these experiments, several conclusions can be drawn:

(a) The selection of the t-norm playing the role of the conjunctive operator does not seem to be of significative importance in fuzzy control because the five continuous t-norms studied have given similar results.

(b) On the other hand, the selection of the implication operator is the main problem of this area, as can be viewed taking into account the results corresponding to the Measure of Convergence. At the sight of the results contained in our study, it can be drawn that the implications being an extension of the boolean conjunction present better behavior in this fuzzy control role than the other ones extending the boolean implication. We can remark that the implication operators not belonging to any of the two families do not have good accuracy as well.

(c) Regarding the Defuzzification Interface, the results seem to help the assertion enunciated by us in [7], remarking that the matching was the best characteristic for the defuzzification in fuzzy control. The average data obtained by the four methods employing it working in mode B go beyond the other two methods, Middle of Maxima and Center of Gravity, working both in mode A and mode B. Hence, two assumptions become clear: mode B is more accurate than mode A and the use of additional information in the defuzzification process (in this case, the consideration of the control rules matching) gives better results.

Anyway, several concrete combinations employing defuzzification methods working in mode A and implication operators not belonging to t-norms family, have demonstrated very good accuracy in different measures in the two applications. As we have remarked in Section 8, FLCs using the R-implications I4, I5 and I27, the S-implication I22 and the operator I2 as implication operators and working in mode A with the Minimum as aggregation operator and the defuzzifier Middle of Maxima (D1a) have a remarkable accuracy in both applications, although the use of these operators in combination with the remaining ones playing the other fuzzy control roles do not have an accurate behavior.

Thus there are two different ways in which to design an FLC to solve a problem:

1. On the one hand, the accuracy of several concrete combinations of operators not belonging to the best families but working well jointly can be taken into account and the controller can be built using them. This selection could be problematic because, although these FLCs have proven good behavior in several experiments (see [5, 20] and this paper), they could be inaccurate in other ones (in [6], some of them do not present good behavior in the fuzzy modeling of $Y = X$, when the Data Bases considered employ a different number of linguistic terms in the primary fuzzy partition).

2. On the other hand, the design can be developed by taking an operator belonging to the best average family for each fuzzy control role. The controller obtained in this way will work well for many kinds of applications and will be adequately accurate (for example, in Section 8 it is pointed out that when the t-norms work as implication operators in the Inverted Pendulum problem, the controller only loses the control of the system in a few cases although they are not the most accurate operators in this role) and its behavior would be performed in the concrete one studied by using some Knowledge Base tuning method (see [17, 15]). In this way, this seems to be the best design option. The Inference System of this FLC will be composed by an implication extending the boolean conjunction as implication operator and any continuous t-norm as conjunctive operator, while its Defuzzification Interface will work in mode B using a defuzzifier based on the matching.

Appendix A. Implication operators selected

$$(\mathrm{I1})\ (x,y) = \begin{cases} 1 & \text{if } x \neq 1 \text{ or } y = 1 \\ 0 & \text{if } x = 1 \text{ and } y \neq 1 \end{cases}$$

$$(\mathrm{I2})\ (x,y) = \begin{cases} 1 & \text{if } x \leqslant y \\ 0 & \text{otherwise} \end{cases}$$

$$(\mathrm{I3})\ (x,y) = \begin{cases} 1 & \text{if } x = y \\ y & \text{otherwise} \end{cases}$$

$$(\mathrm{I4})\ (x,y) = \begin{cases} \mathrm{Min}(1, y/x) & \text{if } x \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

$$(\mathrm{I5})\ (x,y) = \mathrm{Min}(1, 1 - x + y)$$

$$(\mathrm{I6})\ (x,y) = \mathrm{Max}(1 - x, y)$$

$$(\mathrm{I7})\ (x,y) = \mathrm{Max}(1 - x, \mathrm{Min}(x, y))$$

(I8) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(x, y)$

(I9) $(\boldsymbol{x},\boldsymbol{y}) = \text{Max}(\text{Max}(\text{Min}(x, y), 1 - y),$

$\text{Min}(y, 1 - x))$

(I10) $$(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} x & \text{if } y = 1 \\ y & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

(I11) $$(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} x + y & \text{if } x + y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

(I12) $(\boldsymbol{x},\boldsymbol{y}) = \text{Max}(0, y - x)$

(I13) $$(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} y - x & \text{if } y \geqslant x \\ 1 & \text{otherwise} \end{cases}$$

(I14) $$(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 1 & \text{if } y \leqslant x \\ y & \text{otherwise} \end{cases}$$

(I15) $(\boldsymbol{x},\boldsymbol{y}) =$

$$\begin{cases} 1 & \text{if } x = 0 \text{ or } 1 - y = 0 \\ \text{Min}\left(1, \dfrac{y}{x}, \dfrac{1 - y}{1 - x}\right) & \text{if } x > 0 \text{ and } 1 - y > 0 \end{cases}$$

(I16) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(\text{I2}(x, y), \text{ I27}(1 - x, 1 - y))$

(I17) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(\text{I27}(x, y), \text{ I27}(1 - x, 1 - y))$

(I18) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(\text{I27}(x, y), \text{ I2}(1 - x, 1 - y))$

(I19) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(\text{I2}(x, y), \text{ I2}(1 - x, 1 - y))$

(I20) $(\boldsymbol{x},\boldsymbol{y}) = \text{Max}(1 - x, 1 - y)$

(I21) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(1 - x, 1 - y)$

(I22) $(\boldsymbol{x},\boldsymbol{y}) = 1 - x + x \cdot y$

(I23) $(\boldsymbol{x},\boldsymbol{y}) = x + y - x \cdot y$

(I24) $(\boldsymbol{x},\boldsymbol{y}) = |x - y|$

(I25) $(\boldsymbol{x},\boldsymbol{y}) = x \cdot y$

(I26) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}\{\text{Min}(\text{Max}(1 - x, y),$

$\text{Max}(x, 1 - y)), \text{Max}(y, 1 - y)\}$

(I27) $$(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise} \end{cases}$$

(I28) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}\left(1, \dfrac{y \cdot (1 - x)}{x \cdot (1 - y)}\right)$

(I29) $(\boldsymbol{x},\boldsymbol{y}) = y^x$

(I30) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(1, x + y)$

(I31) $(\boldsymbol{x},\boldsymbol{y}) = \text{Max}(0, x + y - 1)$

(I32) $(\boldsymbol{x},\boldsymbol{y}) = \text{Max}(x, y)$

(I33) $$(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$$

(I34) $(\boldsymbol{x},\boldsymbol{y}) = \text{Max}(0, x - y)$

(I35) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(x, 1 - y)$

(I36) $(\boldsymbol{x},\boldsymbol{y}) = \text{Min}(\text{Min}(x, 1 - y), \text{Min}(y, 1 - x))$

(I37) $(\boldsymbol{x},\boldsymbol{y}) = \text{Max}(\text{Min}(x, 1 - y), \text{Min}(y, 1 - x))$

(I38) $(\boldsymbol{x},\boldsymbol{y}) =$

$$\begin{cases} 1 & \text{if } x = 0 \text{ or } 1 - y = 0 \\ \text{Min}\left(1, \dfrac{y}{x}, \dfrac{1 - x}{1 - y}\right) & \text{if } x > 0 \text{ and } 1 - y > 0 \end{cases}$$

(I39) $$(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 1 - x & \text{if } y = 0 \\ y & \text{if } x = 1 \\ 1 & \text{otherwise} \end{cases}$$

(I40) $(\boldsymbol{x},\boldsymbol{y}) = \dfrac{x \cdot y}{x + y - x \cdot y}$

(I41) $(\boldsymbol{x},\boldsymbol{y}) = \dfrac{x \cdot y}{1 + (1 - x) \cdot (1 - y)}$

Appendix B. Study of the properties verified by the implication operators selected

	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12	P13
I1	x	x	x		x								
I2	x	x	x									x	
I3				x	x					x			
I4	x	x	x	x	x							x	
I5	x	x	x	x	x							x	
I6	x	x	x	x	x							x	
I7	x	x	x	x	x							x	
I8		x		x	x	x	x	x	x				x
I9					x	x							
I10		x		x	x	x	x	x	x				x
I11							x			x			
I12	x	x			x					x			
I13										x	x		
I14						x		x			x		
I15				x									
I16	x					x					x		
I17				x	x			x					
I18				x		x							
I19	x	x				x	x						
I20	x		x				x					x	
I21	x						x						
I22	x	x	x	x	x							x	
I23		x			x	x	x	x		x	x		
I24						x	x			x			
I25		x		x	x	x	x	x	x				x
I26				x	x	x	x						
I27	x	x	x	x	x							x	
I28	x	x	x										
I29	x	x	x	x				x					
I30		x			x	x	x	x		x	x		
I31		x		x	x	x	x	x	x				x
I32		x			x	x	x	x		x	x		
I33		x			x	x	x	x		x	x		
I34		x				x			x				
I35						x			x				
I36					x		x		x				
I37					x		x			x			
I38	x	x	x										
I39	x	x	x	x	x							x	
I40		x		x	x	x	x	x	x				x
I41		x		x	x	x	x	x	x				x

Appendix C. Individual application results

Table 8
AD_SE values of the FLCs in the fuzzy modelling of the function $Y = X$ problem

	Mode A (Min)		Mode A (Max)		Mode B					
	D1a	D2a	D1a	D2a	D3b	D6b	D9b	D12b	D13b	D15b
I1	*	*	*	*	0.69315	0.69315	0.69365	0.69365	0.65584	0.63742
I2	*	*	*	*	0.99584	0.99584	0.98277	0.98277	0.82723	0.69158
I4	**1.00000**	0.99196	0.63458	0.63458	0.98722	0.99584	0.98099	0.98277	0.82723	0.71367
I5	**1.00000**	0.86266	0.63458	0.63458	0.85048	0.99584	0.87289	0.98277	0.82723	0.67061
I6	0.96272	0.82234	0.63458	0.63458	0.81576	0.99826	0.83969	0.97580	0.83919	0.65608
I7	0.96476	0.82225	0.63458	0.63458	0.79991	0.94459	0.82387	0.96068	0.75679	0.65141
I8	0.98650	0.98446	0.98653	0.96853	0.98722	0.99584	0.98099	0.98277	0.98211	0.97469
I9	0.63458	0.78430	0.63458	0.63458	0.43223	0.63458	0.40751	0.63458	0.63458	0.65624
I10	*	*	*	*	0.99623	0.99826	0.97377	0.97580	0.98633	0.98582
I11	0.05586	0.03888	0.63458	0.56614	0.39630	0.93072	0.27445	0.92115	0.58161	0.39273
I12	0.63458	0.63458	***0.00000***	0.37907	0.88475	0.88877	0.79541	0.78940	0.60135	0.34946
I13	0.03514	0.03267	0.63458	0.63458	0.56521	0.63458	0.57528	0.63458	0.33479	0.47624
I14	0.63458	0.76416	0.63458	0.63458	0.61559	0.63458	0.62468	0.63458	0.63458	0.66653
I15	**1.00000**	0.99289	0.63458	0.64514	0.98337	0.99584	0.97995	0.98277	0.82723	0.77456
I16	*	*	*	*	0.99424	0.99584	0.98407	0.98277	0.82723	0.74453
I17	0.99961	0.98977	0.63458	0.64534	0.98201	0.99584	0.97867	0.98277	0.82723	0.76142
I18	0.68029	0.67532	0.63458	0.66738	0.98258	0.99705	0.97892	0.98015	0.72561	0.74088
I19	*	*	*	*	0.99584	0.99584	0.98277	0.98277	0.82723	0.87042
I20	0.63458	0.44610	0.63458	0.63458	0.50004	0.63458	0.47320	0.63458	0.63458	0.59985
I21	0.63458	0.65992	0.63458	0.64033	0.53959	0.58365	0.47347	0.50725	0.63458	0.68417
I22	0.99782	0.84416	0.63458	0.63458	0.83438	0.99826	0.85742	0.97580	0.83919	0.66353
I23	0.05622	0.03876	0.63458	0.65553	0.69202	0.93969	0.67087	0.91673	0.60858	0.57083
I24	0.05201	0.02787	0.03514	0.43772	0.48487	0.72498	0.42715	0.63458	0.45086	0.40604
I25	0.94601	0.97396	0.98242	0.97898	0.98858	0.98826	0.98090	0.97580	0.98633	0.98064
I26	0.95585	0.88172	0.70912	0.63897	0.78930	0.94459	0.82387	0.96068	0.75679	0.70714
I27	**1.00000**	0.98855	0.63458	0.63458	0.98851	0.99584	0.98095	0.98277	0.82723	0.70745
I28	0.92603	0.92898	0.63458	0.63458	0.87650	0.87656	0.79188	0.77689	0.80271	0.71084
I29	0.99737	0.99276	0.63458	0.63458	0.98671	0.99826	0.98081	0.97580	0.83919	0.70887
I30	0.05078	0.04019	0.63458	0.67085	0.70665	0.93727	0.67995	0.92370	0.58721	0.60173
I31	0.69338	0.69126	0.98242	0.98929	0.99069	0.99826	0.98000	0.97580	0.98633	0.98909
I32	0.05856	0.03724	0.63458	0.63458	0.67145	0.93969	0.64943	0.91673	0.60858	0.54783
I33	0.05026	0.04029	0.63458	0.66975	0.71846	0.92288	0.69383	0.91902	0.55951	0.61667
I34	0.63458	0.39732	0.63458	0.49366	0.46727	0.63458	0.45391	0.63458	0.63458	0.46813
I35	0.63458	0.49701	0.63458	0.59256	0.56904	0.63458	0.57692	0.63458	0.63458	0.57031
I36	0.92342	0.89696	0.88397	0.83452	0.87195	0.88735	0.79169	0.79637	0.87560	0.83038
I37	0.05309	0.03869	0.03514	0.56563	0.61806	0.71702	0.58016	0.63458	0.43615	0.52170
I38	0.92742	0.92898	0.63458	0.63458	0.87547	0.87878	0.79332	0.79383	0.79629	0.71780
I39	0.98854	0.89259	0.63458	0.63458	0.86764	0.98145	0.89074	0.97810	0.80894	0.68186
I40	0.94601	0.98134	0.98242	0.97173	0.98755	0.99826	0.98090	0.97580	0.98633	0.97621
I41	0.94601	0.97093	0.98242	0.98169	0.98918	0.99826	0.98088	0.97580	0.98633	0.98255

Table 9
Average AD_SE values of the FLCs in the Inverted Pendulum problem

	Mode A (Min)		Mode A (Max)		Mode B					
	D1a	D2a	D1a	D2a	D3b	D6b	D9b	D12b	D13b	D15b
I1	0.72748	0.72748	0.49072	0.49072	0.32216	0.72861	0.72748	0.72748	0.55313	0.49712
I2	**1.00000**	**1.00000**	0.49072	0.49072	0.99733	0.99733	0.96434	0.96434	0.62838	0.51390
I4	**1.00000**	0.98825	0.49072	0.49072	0.99733	0.99733	0.96434	0.96434	0.62838	0.53093
I5	**1.00000**	0.88814	0.49072	0.49072	0.89033	0.99733	0.89111	0.96434	0.62838	0.52081
I6	0.97876	0.86722	0.49072	0.49072	0.87060	0.99733	0.87241	0.96434	0.62838	0.51721
I7	0.95673	0.85005	0.49072	0.49072	0.84953	0.96831	0.85152	0.95230	0.61433	0.51440
I8	0.96943	0.96943	0.96434	0.99773	0.99733	0.99733	0.96434	0.96434	0.96943	0.99855
I9	0.49072	0.60070	0.49072	0.49072	0.15214	0.49072	0.14009	0.49072	0.49072	0.49819
I10	0.84043	0.83630	0.96434	0.99585	0.99733	0.99733	0.96434	0.96434	0.96943	0.99626
I11	0.49072	0.49072	0.49072	0.34619	0.02644	0.75945	***0.00000***	0.72758	0.41879	0.20727
I12	0.49072	0.49072	0.49072	0.35739	0.68981	0.68981	0.72758	0.72758	0.40587	0.31886
I13	0.49072	0.49072	0.49072	0.45375	0.42591	0.49072	0.42373	0.49072	0.32532	0.36058
I14	0.49072	0.65951	0.49072	0.49072	0.48401	0.49072	0.48496	0.49072	0.49072	0.50965
I15	0.98350	0.98219	0.49072	0.49072	0.99733	0.99733	0.96434	0.96434	0.62838	0.57102
I16	0.83750	0.82695	0.49072	0.49072	0.73126	0.73126	0.67940	0.67940	0.62256	0.55243
I17	0.98537	0.98185	0.49072	0.49072	0.99733	0.99733	0.96434	0.96434	0.62838	0.56677
I18	0.49072	0.49072	0.49072	0.49072	0.99733	0.99733	0.96434	0.96434	0.49072	0.49072
I19	*	*	*	*	0.73126	0.73126	0.67940	0.67940	0.62256	0.59216
I20	0.49072	0.22819	0.49072	0.49072	0.24627	0.49072	0.23532	0.49072	0.49072	0.46211
I21	0.49072	0.66038	0.49072	0.49072	0.41294	0.49072	0.42290	0.49072	0.49072	0.52976
I22	0.99565	0.87871	0.49072	0.49072	0.88120	0.99733	0.88254	0.96434	0.62838	0.51903
I23	0.49072	0.49072	0.49072	0.48762	0.52921	0.75945	0.52497	0.72758	0.41879	0.40626
I24	0.49072	0.49072	0.49072	0.34741	0.24269	0.51975	0.22630	0.50276	0.34782	0.27810
I25	0.87502	0.94053	0.96434	0.99573	0.99733	0.99733	0.96434	0.96434	0.96943	0.99733
I26	0.85575	0.87806	0.49072	0.49072	0.84526	0.96831	0.84995	0.95230	0.61433	0.55997
I27	**1.00000**	0.98625	0.49072	0.49072	0.99733	0.99733	0.96434	0.96434	0.62838	0.52914
I28	0.63170	0.67603	0.49072	0.49072	0.67647	0.62718	0.72034	0.69362	0.56429	0.51666
I29	0.98167	0.97882	0.49072	0.49072	0.99733	0.99733	0.96434	0.96434	0.62838	0.52981
I30	0.49072	0.49072	0.49072	0.49211	0.53587	0.75945	0.53069	0.72758	0.41879	0.41351
I31	0.86953	0.87267	0.96434	0.99102	0.99733	0.99733	0.96434	0.96434	0.96943	0.99246
I32	0.49072	0.49072	0.49072	0.48160	0.51786	0.75945	0.51360	0.72758	0.41879	0.39936
I33	0.49072	0.49072	0.49072	0.49378	0.54533	0.75945	0.54017	0.72758	0.41879	0.41973
I34	0.49072	0.20286	0.49072	0.22444	0.21646	0.49072	0.20605	0.49072	0.49072	0.21646
I35	0.49072	0.27273	0.49072	0.29531	0.28762	0.49072	0.28300	0.49072	0.49072	0.28764
I36	0.67337	0.68524	0.61152	0.63947	0.68981	0.68981	0.72758	0.72758	0.68806	0.66372
I37	0.49072	0.49072	0.49072	0.36901	0.32214	0.51975	0.31043	0.50276	0.34782	0.31448
I38	0.68576	0.68721	0.49072	0.49072	0.68981	0.68981	0.72758	0.72758	0.56588	0.51936
I39	0.99583	0.90524	0.49072	0.49072	0.90626	0.99733	0.90599	0.96434	0.62838	0.52615
I40	0.88172	0.96466	0.96434	0.99741	0.99733	0.99733	0.96434	0.96434	0.96943	0.99850
I41	0.87646	0.93180	0.96434	0.99512	0.99733	0.99733	0.96434	0.96434	0.96943	0.99681

Table 10
Average AD_MC values of the FLCs in the Inverted Pendulum problem

	Mode A (Min)		Mode A (Max)		Mode B					
	D1a	D2a	D1a	D2a	D3b	D6b	D9b	D12b	D13b	D15b
I1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I2	0.00000	0.00000	0.00000	0.00000	0.65076	0.65076	0.50001	0.50001	0.00000	0.00000
I4	0.00000	0.99178	0.00000	0.00000	0.65076	0.65076	0.50007	0.50001	0.00000	0.00000
I5	0.00000	0.00000	0.00000	0.00000	0.00000	0.65076	0.00000	0.50001	0.00000	0.00000
I6	0.56477	0.00000	0.00000	0.00000	0.00000	0.65076	0.00000	0.50001	0.00000	0.00000
I7	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I8	0.74918	0.74919	0.50001	0.71843	0.65076	0.65076	0.50009	0.50001	0.74918	0.80628
I9	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I10	0.00000	0.00000	0.50001	0.58906	0.30152	0.65076	0.50009	0.50001	0.74918	0.59714
I11	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I12	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I13	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I14	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I15	0.00000	0.77259	0.00000	0.00000	0.65076	0.65076	0.50000	0.50001	0.00000	0.00000
I16	0.90718	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I17	0.76199	0.75426	0.00000	0.00000	0.65076	0.65076	0.50000	0.50001	0.00000	0.00000
I18	0.00000	0.00000	0.00000	0.00000	0.65076	0.65076	0.50000	0.50001	0.00000	0.00000
I19	*	*	*	*	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I21	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I22	1.00000	0.00000	0.00000	0.00000	0.00000	0.65076	0.00000	0.50001	0.00000	0.00000
I23	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I24	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I25	0.00000	0.00000	0.50001	0.68163	0.65076	0.65076	0.50001	0.50001	0.74918	0.65076
I26	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I27	0.00000	0.95056	0.00000	0.00000	0.65076	0.65076	0.50000	0.50001	0.00000	0.00000
I28	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I29	0.73278	0.00000	0.00000	0.00000	0.65076	0.65076	0.50001	0.50001	0.00000	0.00000
I30	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I31	0.00000	0.00000	0.50001	0.73817	0.65076	0.65076	0.50001	0.50001	0.74918	0.73498
I32	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I33	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I34	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I35	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I36	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I37	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I38	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I39	0.74918	0.00000	0.00000	0.00000	0.00000	0.65076	0.00000	0.50001	0.00000	0.00000
I40	0.00000	0.00000	0.50001	0.00000	0.65081	0.65076	0.50003	0.50001	0.74918	0.74001
I41	0.00000	0.00000	0.50001	0.71464	0.65072	0.65076	0.50001	0.50001	0.74918	0.00000

Table 11
Conjunctive Adaptation Degree values of the FLCs in the Inverted Pendulum problem

	Mode A (Min)		Mode A (Max)		Mode B					
	D1a	D2a	D1a	D2a	D3b	D6b	D9b	D12b	D13b	D15b
I1	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I2	0.00000	0.00000	0.00000	0.00000	0.82405	0.82405	0.73217	0.73217	0.00000	0.00000
I4	0.00000	0.99001	0.00000	0.00000	0.82405	0.82405	0.73217	0.73217	0.00000	0.00000
I5	0.00000	0.00000	0.00000	0.00000	0.00000	0.82405	0.00000	0.73217	0.00000	0.00000
I6	0.77177	0.00000	0.00000	0.00000	0.00000	0.82405	0.00000	0.73217	0.00000	0.00000
I7	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I8	0.85931	0.85931	0.73217	0.85808	0.82405	0.82405	0.73217	0.73217	0.85931	0.90241
I9	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I10	0.00000	0.00000	0.73217	0.79246	0.82405	0.82405	0.73217	0.73217	0.85931	0.79670
I11	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I12	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I13	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I14	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I15	0.00000	0.87739	0.00000	0.00000	0.82405	0.82405	0.73217	0.73217	0.00000	0.00000
I16	0.87234	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I17	0.87368	0.86806	0.00000	0.00000	0.82405	0.82405	0.73217	0.73217	0.00000	0.00000
I18	0.00000	0.00000	0.00000	0.00000	0.82405	0.82405	0.73217	0.73217	0.00000	0.00000
I19	*	*	*	*	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I20	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I21	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I22	0.99782	0.00000	0.00000	0.00000	0.00000	0.82405	0.00000	0.73217	0.00000	0.00000
I23	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I24	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I25	0.00000	0.00000	0.73217	0.83868	0.82405	0.82405	0.73217	0.73217	0.85931	0.82405
I26	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I27	0.00000	0.96841	0.00000	0.00000	0.82405	0.82405	0.73217	0.73217	0.00000	0.00000
I28	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I29	0.85722	0.00000	0.00000	0.00000	0.82405	0.82405	0.73217	0.73217	0.00000	0.00000
I30	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I31	0.00000	0.00000	0.73217	0.86459	0.82405	0.82405	0.73217	0.73217	0.85931	0.86372
I32	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I33	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I34	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I35	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I36	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I37	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I38	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
I39	0.87251	0.00000	0.00000	0.00000	0.00000	0.82405	0.00000	0.73217	0.00000	0.00000
I40	0.00000	0.00000	0.73217	0.00000	0.82407	0.82405	0.73218	0.73217	0.85931	0.86925
I41	0.00000	0.00000	0.73217	0.85488	0.82403	0.82405	0.73217	0.73217	0.85931	0.00000

References

[1] C. Alsina, E. Trillas and L. Valverde, On some logical connectives for fuzzy set theory, *J. Math. Anal. Appl.* **93** (1981) 15–26.

[2] S. Baglio, L. Fortuna, S. Graziani and G. Muscato, Membership function shape and the dynamical behavior of a fuzzy system, *Proc. EUFIT '93* (1993) 645–650.

[3] H. Berenji, Fuzzy logic controllers, in: R.R. Yager and L.A. Zadeh, Eds., *An Introduction to Fuzzy Logic Applications in Intelligent Systems* (Kluwer Academic, Dordrecht, 1992) 69–96.

[4] P.P. Bonissone, Fuzzy logic controllers: an industrial reality, in: J.M. Zurada, R.J. Marks II and C.J. Robinson, Eds., *Computational Intelligence: Imitating Life* (IEEE Press, New York, 1994) 316–327.

[5] Z. Cao and A. Kandel, Applicability of some fuzzy implication operators, *Fuzzy Sets and Systems* **31** (1989) 151–186.

[6] Z. Cao, D. Park and A. Kandel, Investigations on the applicability of fuzzy inference, *Fuzzy Sets and Systems* **49** (1992) 151–169.

[7] E. Cárdenas, J.C. Castillo, O. Cordón, F. Herrera and A. Peregrín, Influence of fuzzy implication functions and defuzzification methods in fuzzy control, *Busefal* **57** (1994) 69–79.

[8] E. Cárdenas, J.C. Castillo, O. Cordón, F. Herrera and A. Peregrín, Applicability of T-norms in fuzzy control, *Busefal* **61** (1995) 28–37.

[9] T.C. Chang, K. Hasegawa and C.W. Ibbs, The effects of membership function of fuzzy reasoning, *Fuzzy Sets and Systems* **44** (1991) 169–186.

[10] O. Cordón, F. Herrera and A. Peregrín, T-norms vs. implication functions as implication operators in fuzzy control, *Proc. 6th IFSA Congr.* (1995) 501–504.

[11] Ch. Dujet and N. Vincent, Force implication: a new approach to human reasoning, *Fuzzy Sets and Systems* **69** (1995) 53–63.

[12] M.M. Gupta and J. Qi, Theory of T-norms and fuzzy inference methods, *Fuzzy Sets and Systems* **40** (1991) 431–450.

[13] M.M. Gupta and J. Qi, Design of fuzzy logic controllers based on generalized T-operators, *Fuzzy Sets and Systems* **40** (1991) 473–489.

[14] H. Hellendoorn, Design and development of fuzzy systems at Siemens R&D, *Proc. FUZZ-IEEE '93* (San Francisco, 1993) 1365–1370.

[15] H. Hellendoorn, D. Driankov and M. Reinfrank, *An Introduction to Fuzzy Control* (Springer, Berlin, 1993).

[16] H. Hellendoorn and C. Thomas, Defuzzification in fuzzy controllers, *J. Intelligent Fuzzy Systems* **1** (1993) 109–123.

[17] F. Herrera, M. Lozano and J.L. Verdegay, Tuning fuzzy logic controllers by genetic algorithms, *Internat. J. Approx. Reasoning* **12** (1995) 293–315.

[18] K. Hirota, *Industrial Applications of Fuzzy Technology* (Springer, Berlin, 1993).

[19] R. Jager, H.B. Verbruggen and P. Bruijn, The role of defuzzification methods in the applications of fuzzy control, *Preprints of SICICA'92* (Malaga, 1992) 111–116.

[20] J. Kiszka, M. Kochanska and D. Sliwinska, The influence of some fuzzy implication operators on the accuracy of a fuzzy model – Parts I and II, *Fuzzy Sets and Systems* **15** (1985) 111–128, 223–240.

[21] C.C. Lee, Fuzzy logic in control systems: fuzzy logic controller – Parts I, II, *IEEE Trans. Systems Man Cybernet.* **20** (1990) 404–418, 419–435.

[22] E. Lembessis and R. Tanscheit, The influence of implication operators and defuzzification methods on the deterministic output of a fuzzy rule-based controller, *Proc. 4th IFSA Congr.* (1991) 109–114.

[23] C. Liaw and J. Wang, Design and implementation of a fuzzy controller for a high performance induction motor drive, *IEEE Trans. Systems Man Cybernet.* **21** (1991) 921–929.

[24] P. Magrez and P. Smets, Fuzzy modus ponens: a new model suitable for applications in knowledge-based systems, *Internat. J. Intelligent Systems* **4** (1989) 181–200.

[25] E.H. Mamdani, Twenty years of fuzzy control: experiences gained and lessons learnt, in: R.J. Marks II, Ed., *Fuzzy Logic Technology and Applications* (IEEE Press, New York, 1993) 19–24.

[26] E.H. Mamdani and S. Assilian, An experiment in linguistic synthesis with a fuzzy logic controller, *Internat. J. Man–Machine Stud.* **7** (1975) 1–13.

[27] M. Mizumoto, Fuzzy controls under various approximate reasoning methods, *Preprints of 2nd IFSA Congr.* (Tokyo, 1987) 143–146.

[28] M. Mizumoto, Pictorial representations of fuzzy connectives, Part I: cases of t-norms, t-conorms and averaging operators, *Fuzzy Sets and Systems* **31** (1989) 217–242.

[29] M. Mizumoto and H.J. Zimmerman, Comparison of fuzzy reasoning methods, *Fuzzy Sets and Systems* **8** (1982) 253–283.

[30] K. Nishimori, H. Tokutaka, K. Fujimura, S. Hirakawa, S. Hamano, S. Kishida and I. Naganori, Comparison of fuzzy reasoning methods on driving control for right or left turning of a model car, *Proc. 5th IFSA Congr.* (1993) 242–245.

[31] N. Pfluger, J. Yen and R. Langari, A defuzzification strategy for a fuzzy logic controller employing prohibitive information in command formulation, *Proc. FUZZY-IEEE'92* (San Diego, 1992) 712–723.

[32] T.A. Runkler and M. Glesner, A set of axioms for defuzzification strategies towards a theory of rational defuzzification operators, *IEEE Internat. Conf. on Fuzzy Systems* (San Francisco, 1993) 1161–1166.

[33] T.A. Runkler and M. Glesner, Defuzzification with improved static and dynamic behavior: extended center of area, *Proc. 1st EUFIT* (1993) 845–851.

[34] E. Trillas and L. Valverde, On implication and indistinguishability in the setting of fuzzy logic, in: J. Kacpryzk and R.R. Yager, Eds., *Management Decision Support Systems Using Fuzzy Sets and Possibility Theory* (Verlag TÜV Rheinland, Köln, 1985) 198–212.

[35] T. Tsukamoto, An approach to fuzzy reasoning method, in: M.M. Gupta, R.K. Ragade and R.R. Yager, Eds., *Advances in Fuzzy Set Theory and Applications* (North-Holland, Amsterdam, 1979).

[36] I.G. Umbers and P.J. King, An analysis of human-decision making in cement kiln control and the implications for automation, *Internat. J. Man–Machine Stud.* **12** (1980) 11–23.

[37] J. Villar and M.A. Sanz, Analyzing the characteristics of the implication functions according to their behaviour in a fuzzy system, *Proc. 4th Spanish Conf. on Fuzzy Logic and Technologies* (Blanes, 1994) 59–64 (in Spanish).

[38] R.R. Yager, Fuzzy sets and approximate reasoning in decision and control, *Proc. IEEE Internat. Conf. in Fuzzy Systems* (1992) 418–428.

[39] R.R. Yager and D.P. Filev, On the issue of defuzzification and selection based on a fuzzy set, *Fuzzy Sets and Systems* **55** (1993) 255–271.

[40] T. Yamakawa, Stabilization of an inverted pendulum by a high-speed fuzzy logic controller hardware system, *Fuzzy Sets and Systems* **32** (1989) 161–180.

[41] L.A. Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338–353.