APPLICABILITY OF T-NORMS IN FUZZY CONTROL

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ABSTRACT

The purpose of this paper is to study the influence of triangular norms in fuzzy logic controllers. In order to make that, we present the generical expression of the Compositional Rule of Inference used in fuzzy control and show the different ways in that t-norms can affect to it. The most important conclusion that can be drawn is that if we use t-norms to make inference we will get fuzzy logic controllers which present better performance than using implication functions.

1.- INTRODUCTION TO FUZZY CONTROL

An important component of a fuzzy logic controller (FLC) is the **Inference Mechanism**, which receives the inputs of the state variables of the controlled system and determines the control action to be applied to it giving values to its control variables. The Inference Mechanism takes the inputs $A' = (A_1 , \ldots , A_n)$, given by the **Fuzzification Interface**, and the Compositional Rule of Inference is applied to them obtaining the fuzzy set B'. Its generical expression is the following:

$$B'(y) = \operatorname*{Sup}_{x} \{ T'(A'(x), I(A(x), B(y))), x \in R^n \}$$

being:
- $A'(x) = T (A_1 '(x_1), A_2 '(x_2), \ldots , A_n '(x_n))$
- $A(x) = T (A_1 (x_1), A_2 (x_2), \ldots , A_n (x_n))$
- I is an inference function
- T' is a connective or a conjunctive operator

Due to the input x, that corresponds to the state variables of the controlled system, is a crisp value in fuzzy control, $x = x_0$, the function A'(x) takes the following expression:

$$A'(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{otherwise} \end{cases}$$

In this way, the expression of the Compositional Rule of Inference is reduced to:

$$B'(y) = T'(1, I (A(x_0), B(y))) = I (A(x_0), B(y))$$

Consequently, any Inference Mechanism is determined by the inference function I and the t-norm T (conjunctive operator used to obtain $A(x_0)$).

A partial control action represented by the fuzzy set B'_i (y) is obtained from every rule R_i of the Knowledge Base applying the Compositional Rule of Inference on this rule. The **Defuzzification Interface** obtains the global control action aggregating this fuzzy sets into one, $B'(y) = \cup_i B_i(y)$, and applying a defuzzification method D that converts this fuzzy set in crisp values associated to the control variables of the system. Thus, denoting by S the FLC, by x_0 the value of the inputs and by y_0 the crisp value obtained by means of the defuzzification process, it is represented by the expression:

$$y_0 = S(x_0) = D(B'(y))$$

2.- APPLICABILITY OF T-NORMS IN FUZZY CONTROL

As we have mentioned in the previous section, the Inference Mechanism of a FLC depends directly on two agents, the conjunctive operator and the inference function both used to design the Compositional Rule of Inference. T-norms can take a role in that both aspects in fuzzy control (see [3,4,5,6]):

1. Due to the conjunctive operator that aggregates the values given by every input to the system is, in fact, a t-norm, the output of the FLC will depend directly on the choice of this operator.

2. Furthermore the influence of t-norms on inference process is presented in a double way:

   2.1. T-norms are important elements in the design of implication functions (see [2,4,5]), as we show in the following:

   - **Residual Implications or R-implications**: Defined by means of a t-norm T in the following way $I(x,y) = \text{Sup} \{ c : c \in [0,1] / T(c,x) \leq y \}$
   - **Quantum Mechanics Implications or QM-implications**: Belonging to this family the implication functions that presents the form $I(x,y) = S(N(x), T(x,y))$, being S a t-conorm, N a negation function and T a t-norm.

   2.2. On the other hand, it is possible to use a t-norm as an inference function to make inference in fuzzy control (see [3,4]).

In order to develop this work, six of the more usual t-norms have been selected to be applied like conjunctive operators as well as inference functions. Furthermore we compare them with three usual R-implications, built using three of the t-norms previously selected. In [1,2] was presented a comparative study of the implication functions behavior in fuzzy control, showing the R-implications best performance.

3.- T-NORMS AND R-IMPLICATIONS SELECTED

We have chosen the following six t-norms:

T1-I1.- Logical Product:	**T2-I2.- Hamacher Product:**	**T3-I3.- Algebraic Product:**
$T(x,y) = \mathrm{Min}(x, y)$	$T(x,y) = \dfrac{x \cdot y}{x + y - x \cdot y}$	$T(x,y) = x \cdot y$

T4-I4.- Einstein Product:	**T5-I5.- Bounded Product:**	**T6-I6.- Drastic Product:**
$T(x,y) = \dfrac{x \cdot y}{1 + (1-x) + (1-y)}$	$T(x,y) = \mathrm{Max}(0, x+y-1)$	$T(x,y) = \begin{cases} x & \text{si } y = 1 \\ y & \text{si } x = 1 \\ 0 & \text{en otro caso} \end{cases}$

By means of the generical expression of the R-implications and the t-norms T1, T3 and T6, respectively, the following implication functions are obtained:

I7.- Gödel Implication:	**I8.- Göguen** Implication:	**I9.- Gaines** Implication:
$I(x,y) = \begin{cases} 1 & \text{si } x \leq y \\ y & \text{si } x > 0 \end{cases}$	$I(x,y) = \begin{cases} \mathrm{Max}(1, y/x) & \text{si } x \neq 0 \\ 1 & \text{si } x = 0 \end{cases}$	$I(x,y) = \begin{cases} 1 & \text{si } x \leq y \\ 0 & \text{en otro caso} \end{cases}$

Due to we are going to use this nine operators to make inference in fuzzy control, we will call them generically in the following **inference operators** or **inference functions.**

Next, we present graphically the behavior of the inference operators selected in FLC's. We suppose that the fuzzy set B, associated to the consequent of the rule of the Knowledge Base, is a trapezoidal fuzzy number characterized by the values (x_0, x_1, x_2, x_3). Figure 1 shows its graphical representation.

Let $h = A(x_0) = T(A_1(x_1), A_2(x_2), \ldots, A_n(x_n))$ be the value obtained by applying the conjunctive operator to the inputs, the fuzzy set B' obtained making inference looks like the following figures shown. We have to remark that the dotted line draws the previous fuzzy set B and the continuos broad line, the fuzzy set B' obtained by inferring using every one of the selected operators.

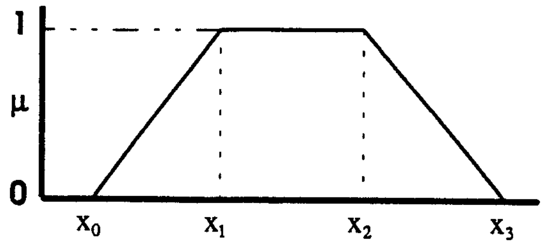


Fig. 1: Graphical representation of the fuzzy set B

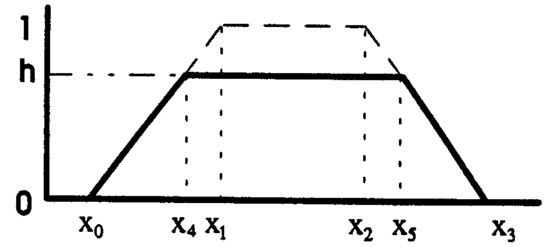


Fig. 2: Fuzzy set B' obtained making inference using t-norm I1

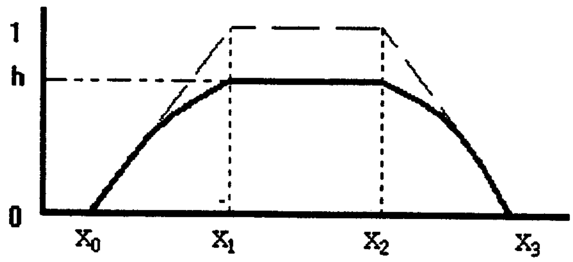


Fig. 3: Fuzzy set B' obtained making inference using t-norm I2

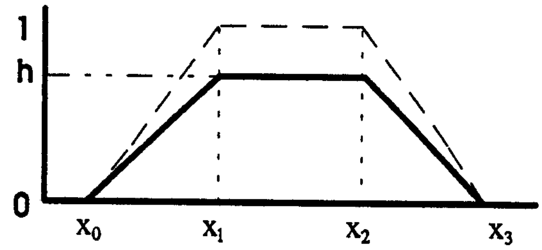


Fig. 4: Fuzzy set B' obtained making inference using t-norm I3

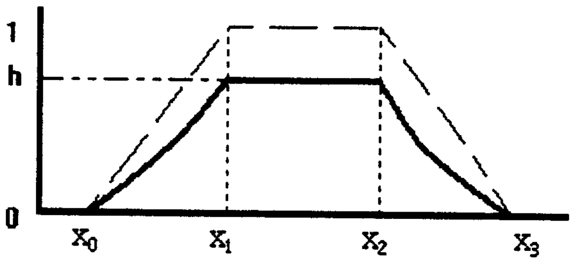


Fig. 5: Fuzzy set B' obtained making inference using t-norm I4

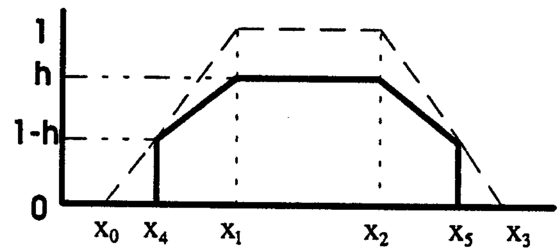


Fig. 6: Fuzzy set B' obtained making inference using t-norm I5

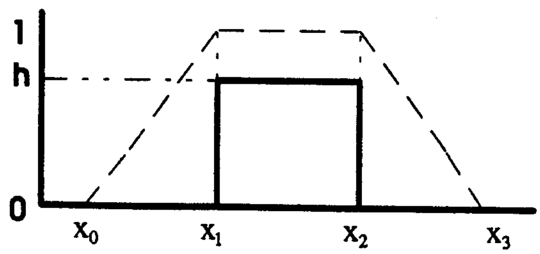


Fig. 7: Fuzzy set B' obtained making inference using t-norm I6

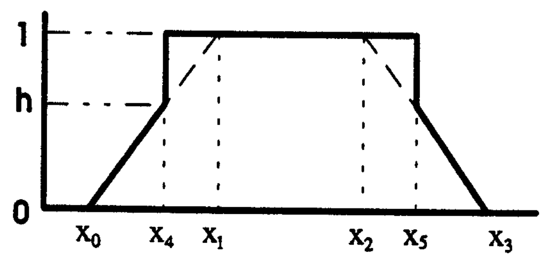


Fig. 8: Fuzzy set B' obtained making inference using R-implication I7

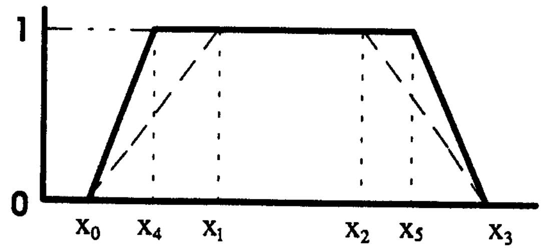


Fig. 9: Fuzzy set B' obtained making inference using R-implication I8

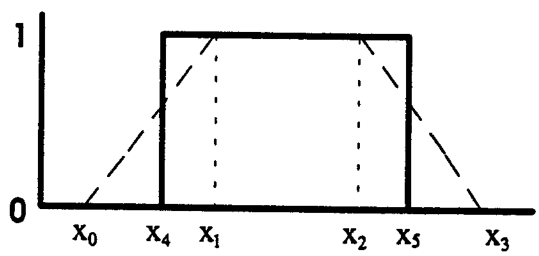


Fig. 10: Fuzzy set B' obtained making inference using R-implication I9

The values x_4 and x_5 are defined by the following expressions: $x_4 = x_0 + (x_1 - x_0)\cdot h$ and $x_5 = x_3 - (x_3 - x_2)\cdot h$.

4.- DEFUZZIFICATION METHODS

On studies developed in [1] and [2] was shown the **Maximum Value weighed by the Matching** like the best defuzzification method of all selected in those works and it was remarked that the defuzzification methods based on weighing present the best performance. In order to develop our study we use the defuzzification methods based on the weighing of the Maximum Value (MV), G, by the three Importance Degrees presented in those papers, the Area, S, the Height, Y, and the Matching, H. The associated expressions are:

D1.- MV weighed by the Area:	**D2.- MV weighed by the Height:**	**D3.- MV weighed by the Matching:**
$y_0 = \frac{\sum_i S_i \cdot G_i}{\sum_i S_i}$	$y_0 = \frac{\sum_i Y_i \cdot G_i}{\sum_i Y_i}$	$y_0 = \frac{\sum_i H_i \cdot G_i}{\sum_i H_i}$

From the aspect of the fuzzy sets obtained making inference by means of the inference operators, shown in a graphical way in the previous section, and the formulation of the defuzzification methods recently presented we can remark the following:

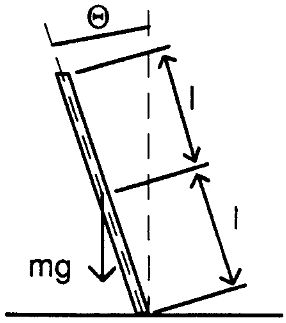
1. The use of t-norms in the role of inference functions to make inference provokes that the Height of the inferred fuzzy set takes the same value that the Matching of the fuzzy sets that constitute the antecedent of the control rule fired. Thus defuzzification methods based on both Importance Degrees (in this work, D2 and D3) will give the same results as making inference using a t-norm.

2. Making inference using the Minimum t-norm (I1) or the R-implications I7, I8 and I9 provokes that the Maximum Value of the fuzzy set obtained take the same value. It is due to Maximum Value is determined by the central values of that fuzzy set, which are equal in the four cases. The same is obtained when using the other five t-norms selected. Both values are different one to one except when the fuzzy sets that constitute the control rules of the Knowledge Base present symmetrical membership functions.

3. Due to defuzzifier D3 depends on the Matching and the Maximum Value, and the Matching does not depend on the inference operator used, this defuzzification method will take the same value when using the four inference functions commented in last point on one hand and when using the other five inference operators on the other hand. Again both values are equal only when symmetrical fuzzy sets are used.

5.- EXPERIMENT SELECTED: THE INVERTED PENDULUM

The chosen application to develop the study of the performance of the different FLC's have been extensiously studied in classical Control Theory, the problem of the **Inverted Pendulum**:



The behaviour of the pendulum is managed by the equation:

$$m\frac{l^2}{3}\frac{d^2\Theta}{dt^2} = \frac{1}{2}\left(-F + m\,g\,\mathrm{sen}\Theta - k\frac{d\Theta}{dt}\right)$$

where $k\frac{d\Theta}{dt}$ is an approximation of the friction strength.

We have worked with the same simulation model of the system that we used to develop the works [1] and [2]. The data used have been M = 5 Kilograms and L = 5 meters and the Knowledge Base is constituted by seven control rules (see [1,2,7]).

6.- MEASURES OF COMPARISON

The two kinds of performance measures, Measures of Convergence and Measures of Error, were presented in [2]. The first family is constituted by measures based on the oscillations of the controlled system around its point of equilibrium. Measures of the second kind are based on the comparison of the behavior of the FLC with a set of evaluation data of the controlled system. We have selected one measure belonging to every family, the **Measure of Convergence** (MC) and the **Measure of Medium Square Error** (SE). Denoting by S [i,j,k] a FLC formed by the t-norm i like conjunctive operator (i = 1, ..., 6), the inference operator j (j = 1, ..., 9) and the defuzzification method k (k = 1, ..., 3), those measures take the following expressions:

$$MC\,(S\,[i,j,k]\,) = \frac{\sum_{ti=m}^{n} |\,e\,(ti)\,|}{(\,n-m\,)\;/\;\Delta t} \qquad SE\,(S\,[i,j,k]\,) = \frac{\frac{1}{2}\sum_{k=1}^{N}(\,y_k - S\,[i,j,k]\,(x_k)\,)^2}{N}$$

being e (t_i) the state of the system in a concrete instant of time t_i, $\Delta t = |t_i - t_{i-1}|$ and m, n the ends of the interval of time in what is done the data recollection in the Measure of Convergence; while (x_k,y_k) is the array of values of the state-control variables belonging to the set of evaluation data of the behavior of the system for the Measure of Medium Square Error.

7.- EXPERIMENTS AND RESULTS

As follows, a table that present the values obtained by the different FLC's in the two performance measures selected is shown. In every one of the cells, the value placed up corresponds to the Measure of Medium Square Error and the value placed down corresponds to the Measure of Convergence.

	T-norms						*R-implications*		
	I1	I2	I3	I4	I5	I6	I7	I8	I9
	Conjunctive Operator: T1.- Logical Product								
D1	9521 **34.518**	11350 56.110	**8941** 117.20	11335 ∞	**8941** 117.20	**8941** 117.19	40431 ∞	161153 ∞	176240 ∞
D2	**8941** 117.18	**8941** 117.19	**8941** 117.19	**8941** 117.19	**8941** 117.19	**8941** 117.19	29700 66.887	148367 ∞	148367 ∞
D3	**8941** 117.18	**8941** 117.19	**8941** 117.19	**8941** 117.19	**8941** 117.19	**8941** 117.19	**8941** 117.18	**8941** 117.18	**8941** 117.18
	Conjunctive Operator: T2.- Hamacher Product								
D1	**8936** 57.437	11360 92.916	9308 47.469	11507 80.393	9308 47.469	9308 47.468	41972 89.548	159817 ∞	173153 ∞
D2	9308 **47.460**	9308 47.468	9308 47.468	9308 47.468	9308 47.468	9308 47.468	29700 65.836	148367 ∞	148367 ∞
D3	9308 **47.460**	9308 47.468	9308 47.468	9308 47.468	9308 47.468	9308 47.468	9308 **47.460**	9308 **47.460**	9308 **47.460**
	Conjunctive Operator: T3.- Algebraic Product								
D1	**9871** 81.327	11561 69.474	10588 74.630	11881 72.119	10588 74.630	10588 74.627	43306 ∞	159315 ∞	172014 ∞
D2	10588 74.588	10588 74.627	10588 74.627	10588 74.627	10588 74.627	10588 74.627	29700 **65.669**	148367 ∞	148367 ∞
D3	10588 74.588	10588 74.627	10588 74.627	10588 74.627	10588 74.627	10588 74.627	10588 74.588	10588 74.588	10588 74.588
	Conjunctive Operator: T4.- Einstein Product								
D1	**10555** 73.692	11799 78.573	11272 77.085	12214 **58.892**	11272 77.085	11272 77.112	43646 ∞	158999 ∞	171314 ∞
D2	11272 77.295	11272 77.096	11272 77.096	11272 77.096	11272 77.096	11272 77.096	29700 65.669	148367 ∞	148367 ∞
D3	11272 77.295	11272 77.096	11272 77.096	11272 77.096	11272 77.096	11272 77.096	11272 77.295	11272 77.295	11272 77.295
	Conjunctive Operator: T5.- Bounded Product								
D1	14998 81.249	14998 80.546	15022 81.335	15030 80.261	15022 81.335	15022 81.337	14985 81.453	158380 ∞	169976 ∞
D2	15022 **79.178**	15022 81.335	15022 81.335	15022 81.335	15022 81.335	15022 81.335	**14964** 83.661	148367 ∞	148367 ∞
D3	15022 **79.178**	15022 81.335	15022 81.335	15022 81.335	15022 81.335	15022 81.335	15022 **79.178**	15022 **79.178**	15022 **79.178**

	Conjunctive Operator: T6.- Drastic Product								
D1	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	153401 ∞	159376 ∞
D2	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	148367 ∞	148367 ∞
D3	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞	**106159** ∞

The two following tables present the medium values obtained by the different conjunctive operators and inference operators in both measures. Point out that the quantity shown into brackets corresponds to the number of combinations Ti-Ij-Dk that lose the control of the system during the interval of time between m and n (MC = ∞):

	T1	T2	T3	T4	T5	T6
EC	**33237**	33360	34258	34755	35492	113006
MC	107.954 (6)	**53.935 (4)**	74.167 (5)	75.7065 (5)	80.887 (4)	∞ (27)

	I1	I2	I3	I4	I5	I6	I7	I8	I9
EC	**26812**	27211	26881	27261	26881	26881	38428	111253	115198
MC	**74.6 (3)**	78.2 (3)	79.5 (3)	77.6 (4)	79.5 (3)	84.9 (3)	76.2 (6)	79.1(13)	79.1(13)

Finally, we attach a table that makes possible to compare the values obtained by the six FLC's that use the same t-norm like conjunctive operator and inference operator with that who obtain the best values in both measures using the same inference operator and another different conjunctive operator. The information shown into brackets corresponds to the defuzzification method and the conjunctive operator (in the second case) used by the FLC that gets this values:

	I1	I2	I3	I4	I5	I6
SE	8941 (D2, D3)	9308 (D2, D3)	10588 (D1, D2, D3)	11272 (D2, D3)	15022 (D1, D2, D3)	106159 (D1, D2, D3)
MC	34.518 (D1)	47.468 (D2, D3)	74.627 (D2, D3)	58.892 (D1)	81.335 (D1, D2, D3)	∞ (D1, D2, D3)
SE	8936 (T2-D1)	8941 (T1-D2,3)	8941 (T1-D1,2,3)	8941 (T1-D2,3)	8941 (T1-D1,2,3)	8941 (T1-D1,2,3)
MC	34.518 (T1-D1)	47.468 (T2-D2,3)	47.468 (T2-D2,3)	47.468 (T2-D2,3)	47.468 (T2-D2,3)	47.468 (T2-D1,2,3)

8.- ANALYSIS OF RESULTS

At the sight of the average results obtained in the Measure of Medium Square Error the best conjunctive operator is **T1**, Minimum t-norm. In the same way, when we use this t-norm in the role of inference function we get the best results at the sight of average values. On the contrary, when the average results corresponding to the Measure of Convergence are studied it is observed that the best connective is t-norm **T2, Hamacher Product**, and the best inference operator is again **I1, Minimum t-norm** when it is used in this role.

The results obtained by the t-norm **T6, Drastic Product**, when it is used like conjunctive operator have made clear that this connective is not very useful in fuzzy control. Probably this bad behavior can be imputed to the fact that this t-norm is not a continuous one.

In relation to the inference operators, the observed results can play false because the average values obtained in the Measure of Convergence by the **Göguen** and **Gaines R-implications, I8** and **I9** respectively, do not seem to be worse than the ones obtained by the other inference operators. The problem is that thirteen of the eighteen combinations that use this two implication functions can not be able to control the system during the interval of time corresponding to the data capture. This fact joined to the bad results presented by this operators in the Measure of Medium Square Error tell us that it is not a good idea to use them in FLC's. Only **Gödel R-implication** presents values that looks like the ones obtained by the different t-norms although there are six FLC's too that loses control of the system when are using this implication function.

In [1] and [2] there is developed a study of the behavior of several inference functions belonging to the three families (S-implications, R-implications and QM-implications (see [5])) in fuzzy control. By means of the results remarked in those works it is possible to draw the conclusion that R-implications are the best family of implication functions to use in FLC's. Now when we join this results to the ones presented in this paper, we can draw that **when t-norms are used to make inference they are the best family of inference functions that can be employed for this task in fuzzy control**, taking account that it is not too much difference in the behavior of them.

As t-norms are not implication functions, they clearly do not satisfy some of the properties of those (see [5]). The differences are placed in the two following properties that are satisfied by the implication functions but do not by the t-norms:

- If $x' \leq x$ then $I(x,y) \leq I(x',y)$
- $I(0,x) = 1$ (Falsity Principle)

When the second property is studied more deeply it is observed that it not seem to be adequate in fuzzy control because it makes a control action to be applied when there is no matching between the inputs given by the controlled system and the rules of the Knowledge Base. Really no action should be applied in this case and in fact it is verified by the t-norms with the property $T(0,x) = 0$.

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