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Optimality for fuzzified mathematical programming problems: A parametric approach*

F. Herrera
Department of Computer Science and Artificial Intelligence, University of Granada, 18071 Granada, Spain

M. Kovács
Computer Center, Eötvös Loránd University, P.O. Box 157, Budapest 112, H-1502, Hungary

J.L. Verdegay
Department of Computer Science and Artificial Intelligence, University of Granada, 18071 Granada, Spain

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Abstract: From a conventional mathematical programming model, and in accordance with which fuzzification is used, several models of fuzzy mathematical programming problems can be obtained. This paper deals with the study of the optimality concept for (g, p)-fuzzified mathematical programming problems. An auxiliary parametric mathematical programming problem is presented which allows the above model to be solved in a straightforward way. In addition, some results about the (g, p)-fuzzified mathematical programming problem are obtained using the parametric mathematical programming problem.

Keywords: Fuzzy mathematical programming; parametric mathematical programming.

1. Introduction

There are many Mathematical Programming (MP) problems that cannot be modeled in a classical way because the different elements of the problem are vaguely defined. A tool to make MP models more realistic and human-consistent, and hence more applicable, is Zadeh's fuzzy sets theory. Thus, Fuzzy Mathematical Programming (FMP) is a tool to deal with this fuzziness which causes difficulties in modeling [1, 2, 4, 7–11, 13]. A survey on approaches, problems and methods of FMP can be found in [3].

In this paper we will examine the fuzzified version of this problem assuming that the coefficients in the problem formulation are given by fuzzy numbers and the relations in the definition of the feasible set are also fuzzy. Specially, we will examine the so called (g, p)-fuzzified mathematical programming problems [6, 7, 8]. In this approach the side function g of the fuzzifying parameters and the generator function g^p of the Archimedean t-norm, used in the extension principle and in the intersection and Cartesian product of fuzzy sets, are defined by the same function g. Thus, an inequality is fuzzified using a generator function, g, of an Archimedean t-norm ($g^p:[0, 1]\to[0, t]$, $g^p(x)=(g(x))^p$). We present an alternative formulation of the optimality concept presented by Kovács [7, 8]. This formulation is based on the parametric programming which has been studied in [1, 2, 11].

Correspondence to: Dr. F. Herrera, Department of Computer Science and Artificial Intelligence, University of Granada, 18071 Granada, Spain.
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Let $g: I \to \boldsymbol{R}_+$ be a fixed function with the properties of a generator function and let $\mathbf{F}_g$ denote the subset of $\mathbf{F}(\mathbf{R})$ (set of fuzzy numbers over $\mathbf{R}$) containing fuzzy numbers with membership functions

$$\mu(a) = \begin{cases} g^{(-1)}(|a-\alpha|/d) & \text{if } d>0, \\ X_{\{\alpha\}}(a) & \text{if } d=0, \end{cases} \tag{1}$$

for all $\alpha \in \mathbf{R}$, $d \in \mathbf{R}_+ \cup \{0\}$. The elements of $\mathbf{F}_g$ will be called quasitriangular fuzzy numbers generated by g with the center α and spread d, and we will denote it by the pair (α, d).

Let T_{gp} be an Archimedean t-norm given by the generator function g^p, $1 \leq p \leq \infty$. It is easy to see that $\lim_{p\to\infty} T_{gp}(a, b) = \min(a, b)$, therefore we will also use the notation T_{gp} in the case $p = \infty$ meaning the min-norm for $T_{g\infty}$.

The T_{gp}-Cartesian product of r quasitriangular fuzzy numbers generated by g will be called (g, p)-fuzzy vector on $\mathbf{F}_g^r$, i.e. $\mu_a = \mu_1 \times \cdots \times \mu_r$, where $\mu_i = (\alpha_i, d_i) \in \mathbf{F}_g$, $i = 1, \ldots, r$. We will use the short notation $\mu_a = (\alpha, d) \in \mathbf{F}_g^r$, meaning $\alpha \in \mathbf{R}^r$. Without lack of generality we can assume that $d_i \neq 0$ for $i = 1, \ldots, k$ and $d_i = 0$ for $i = k+1, \ldots, r$, where the cases $k = 0$ and $k = r$ are also allowed. It is easy to show that

$$\mu_a(a) = \mu_a(a_1, \ldots, a_r) = \begin{cases} g^{(-1)}(\|\hat{D}^{-1}(\hat{a} - \hat{\alpha})\|_p) & \text{if } k = r \text{ or } a_i = \alpha_i,\ i = k+1, \ldots, r, \\ 0 & \text{otherwise,} \end{cases}$$

if $k > 0$, where $\hat{D} = \operatorname{diag}(d_1, \ldots, d_k)$, $\hat{a} = (a_1, \ldots, a_k)$, $\hat{\alpha} = (\alpha_1, \ldots, \alpha_k)$ and

$$\|a\|_p = \begin{cases} \left(\sum_{j=1}^{k} |a_j|^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max_{j=1,\ldots,k} |a_j| & \text{if } p = \infty. \end{cases}$$

If $k = 0$, then it is obvious that

$$\mu_a(a) = \mu_a(a_1, \ldots, a_r) = X_{\{\alpha_1, \ldots, \alpha_r\}}(a_1, \ldots, a_r).$$

Thus, let the inequality $f(a, x) = \sum_{j=1}^r a_j h_j(x) \leq a_0$ be fuzzified, according to [8], by the (g, p)-fuzzy parameter vector $\mu_a = \bar{\mu}_a = \mu_0 \times \mu_a = (\bar{\alpha}, \bar{d}) \in \mathbf{F}_g^{r+1}$, where $\bar{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_r)$, $\bar{d} = (d_0, d_1, \ldots, d_r)$, $d_0 \geq 0$, $d_j > 0$, $j = 1, \ldots, k$ and $d_j = 0$, $j = k+1, \ldots, r$.

Then the (g, p)-valued inequality relation is a fuzzy set $\sigma \in \mathbf{F}(\mathbf{R}^r)$ with the membership function

$$\sigma(x) = \begin{cases} g^{(-1)}(\max\{0, f(\alpha, x) - \alpha_0\}/\|\overline{Dh}(x)\|_q) & \text{if } \overline{Dh}(x) \neq 0, \\ X_{\{x: f(\alpha, x) - \alpha_0 \leq 0\}}(x) & \text{otherwise,} \end{cases} \tag{2}$$

where $\bar{D} = \operatorname{diag}(d_0, d_1, \ldots, d_r)$, $\bar{h}(x) = (-1, h_1(x), \ldots, h_r(x))$,

$$q = \begin{cases} 1 & \text{if } p = \infty, \\ \infty & \text{if } p = 1, \\ p/(p-1) & \text{otherwise,} \end{cases} \quad \text{and} \quad \|a\|_q = \begin{cases} \left(\sum_{j=1}^{r} |a_j|^q\right)^{1/q} & \text{if } 1 \leq q < \infty, \\ \max_{j=1,\ldots,r} |a_j| & \text{if } q = \infty. \end{cases}$$

The fuzzy feasible set is defined by the T_{gq}-intersection of the (g, p)-valued relations of the fuzzified inequalities.

Next, we will focus on the concept of optimality presented by Kovács in [8], and then we will present an alternative formulation based on the use of parametric mathematical programming problems. With this aim, the paper is set up as follows. In Section 2 the above mentioned approach to solve the problem is briefly introduced. Section 3 is devoted to relate this approach with the use of parametric programming problems. Finally, to clarify the above developments a numerical example is analyzed and some conclusions are pointed out.

2. Fuzzification of the mathematical programming problem

Let us consider the classical MP problem

$$\min \quad f_0(\gamma, x) = \sum_{j=1}^{r} \gamma_j h_j(x), \tag{3}$$

$$\text{s.t.} \quad f_i(\alpha_i, x) = \sum_{j=1}^{r} \alpha_{ij} h_j(x) \leq \alpha_{i0}, \quad i \in M = \{1, \ldots, s\},$$

$$x_j \geq 0, \quad j \in N = \{1, \ldots, r\},$$

where $\gamma = (\gamma_1, \ldots, \gamma_r)$ and $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{ir}) \in \mathbf{R}^r$, and $\alpha_{i0} \in \mathbf{R}$, $\forall i \in M$.

Let the i-th relation in (3) be fuzzified by the (g, p)-fuzzy vector $\mu_i = (\bar{\alpha}_i, \bar{d}_i) \in \mathbf{F}_g^{r+1}$ for $i = 1, \ldots, s$, where $\bar{\alpha}_i = (\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{ir})$, $\bar{d}_i = (d_{i0}, d_{i1}, \ldots, d_{ir})$, $d_{ij} = \beta_i \cdot d_j$, $\beta_i > 0$, $i = 1, \ldots, s$, $j = 0, 1, \ldots, r$, $d_0 \geq 0$, $d_j > 0$ for $j = 1, \ldots, k$ and $d_j = 0$ for $j = k+1, \ldots, r$.

Then the (g, p)-fuzzy feasible set is defined as the T_{gq}-intersection of the fuzzified inequalities, according to [8] by the membership function

$$\vartheta_C(x) = \begin{cases} g^{(-1)}(G(x)/\|\overline{Dh}(x)\|_q) & \text{if } \overline{Dh}(x) \neq 0, \\ X_C(x) & \text{otherwise,} \end{cases} \tag{4}$$

where

$$G(x) = \|(\mathcal{A}h(x) - \alpha^0)_+\|_{qB}$$

and $(\mathcal{A}h(x)q - \alpha^0)_+$ denotes the vector in which the i-th coordinate is defined by

$$(f_i(\alpha_i, x) - \alpha_{i0})_+ = \max\{0, f_i(\alpha_i, x) - \alpha_{i0}\},$$

and $\alpha^0 = (\alpha_{10}, \ldots, \alpha_{s0}\}$. $\|\cdot\|_{qB}$ is the weighted l_q-norm with the weight matrix $B = \operatorname{diag}(\beta_1^{-1}, \ldots, \beta_s^{-1})$, $D = \operatorname{diag}(d_0, d_1, \ldots, d_r)$, $\bar{h}(x) = (-1, h_1(x), \ldots, h_r(x))$ and X_C is the characteristic function of the constraint set defined in (3). We are supposing that all constraints are inequalities, in the case of equalities they may be written as inequalities easily.

Introducing the notions $\vartheta^* = \sup_{x \in \mathbf{R}^r} \vartheta_C(x)$ and $C_\vartheta^* = \{x \in \mathbf{R}^r \mid \vartheta_C(x) = \vartheta^*\}$ the following statements are valid:

(a) $\vartheta^* = 1$ and $C_\vartheta^* \neq \emptyset$ if (3) has a solution.

(b) $\vartheta^* = 0$ then there is no consistent perturbation of the constraint set (3).

(c) If $0 < \vartheta^* < 1$ then there is no solution of (3) in a classical sense.

Thus, the concept of optimality was introduced as follows: A fuzzy number $\mu_{00} = (y, d_{00}) \in \mathbf{F}_g$ is a (g, p)-fuzzy aspiration level for the objective function with the optimality rate $\omega(x, y)$, if $\omega(x, y)$ is the (g, p)-valued relation $\leq$ between the fuzzified objective function and the aspiration level.

Let the objective function be fuzzified by the (g, p)-fuzzy vector $\mu_0 = (\gamma, d^0) \in \mathbf{F}_g^r$, $\gamma = (\gamma_1, \ldots, \gamma_r)$ and $d^0 = (d_{01}, \ldots, d_{0r})$. The optimality rate is [8]

$$\omega(x, y) = \begin{cases} g^{(-1)}((f_0(\gamma, x) - y)_+/\|\overline{Dh}(x)\|_q) & \text{if } \|\overline{Dh}(x)\|_q \neq 0, \\ X_{\{x : f_0(\gamma, x) - y \leq 0\}}(x) & \text{otherwise,} \end{cases} \tag{5}$$

where $(f_0(\gamma, x) - y)_+ = \max\{0, f_0(\gamma, x) - y\}$.

The fuzzy optimum set $\omega^*(x, y)$ with the fuzzy aspiration level μ_{00} is the T_{gq} restriction of the fuzzy set on $X \times Y$ defined by the optimality rate to the fuzzy feasible set ϑ_C.

If the fuzzifying vector $\bar{\mu}_0 = \mu_{00} \times \mu_0 = (\bar{\alpha}_0, \bar{d}_0)$ is such that $\bar{\alpha}_0 = (y, \gamma)$, $\bar{d}_0 = (d_{00}, d_{01}, \ldots, d_{0r})$, $d_{0j} = \beta_0 \cdot d_j$, $j = 0, 1, \ldots, r$, $\beta_0 > 0$, and the vector $\bar{d} = (d_0, d_1, \ldots, d_r)$ is the same which was used for the fuzzification of the feasible set, then it is easy to verify [8]

$$\omega^*(x, y) = T_{gq}(\omega(x, y), \vartheta_C(x)) = \begin{cases} g^{(-1)}(\Phi(x, y)/\|\overline{Dh}(x)\|_q) & \text{if } \|\overline{Dh}(x)\|_q \neq 0, \\ X_{\{x : f_0(\gamma, x) - y \leq 0\}}(x) & \text{otherwise,} \end{cases} \tag{6}$$

where $\Phi(x, y) = \|((f_0(\gamma, x) - y)_+/\beta_0, G(x))\|_q$.

Therefore the (g, p)-fuzzy aspiration level $\mu_0^* = (y^*, d_{00})$ is optimal for the fuzzified MP problem if y^* is the minimal root of the equation

$$\Psi(y) = \sup_{x \in \mathbf{R}_+^r} \omega^*(x, y) = \vartheta^*. \tag{7}$$

In the next section we will present a relation between this approach and the parametric mathematical programming problem, which we will build using the above results.

3. The parametric approach

Let us consider in (1) the linear function $g:[0, 1] \to [0, 1]$, $g(x) = 1 - x$, which is the generator function of the Łukasiewicz t-norm. Then we obtain the following membership function:

$$\mu(a) = \begin{cases} \max\{0, 1 - |a - \alpha|/d\} & \text{if } d > 0, \\ 0 & \text{if } d = 0, \end{cases} \tag{8}$$

which is a linear membership function and may be written $\forall \alpha \in \mathbf{R}$ and $\forall d \in \mathbf{R}_+ \cup \{0\}$ as

$$\mu(a) = \begin{cases} 1 - |a - \alpha|/d & \text{if } \alpha - d \leqslant a \leqslant \alpha + d, \\ 0 & \text{otherwise}. \end{cases} \tag{9}$$

If we consider $P = 1$ then $q = \infty$, so $\|x\|_q = \max_{j=1,\dots,r} |x_j|$, and under these conditions it is obtained that

$$G(x) = \|(\mathscr{A}h(x) - \alpha_0\|_{qB} = \max_{i=1,\dots,s} \{(f_i(\alpha_i, x) - \alpha_{i0})_+/\beta_i\}$$

and the $(g, 1)$-fuzzy feasible set is obtained as

$$\vartheta_C(x) = \max\left\{0, 1 - \max_{i=1,\dots,s} ((f_i(\alpha_i, x) - \alpha_{i0})_+/(\beta_i \|\overline{Dh}(x)\|_q))\right\} \tag{10}$$

which can be written as

$$\vartheta_C(x) = \begin{cases} 1 - \max_i ((f_i(\alpha_i, x) - \alpha_{i0})_+/(\beta_i \|\overline{Dh}(x)\|_q)) & \text{if } 0 \leqslant \max_i (f_i(\alpha_i, x) - \alpha_{i0})_+ \leqslant \beta_i \|\overline{Dh}(x)\|_q, \\ 0 & \text{otherwise}. \end{cases} \tag{11}$$

Let us modify this membership function of the fuzzy feasible set as follows:

$$\tilde{\vartheta}_C(x) = \begin{cases} 1 & \text{if } f_i(\alpha_i, x) \leqslant \alpha_{i0}\ \forall i, \\ 1 - \max_{i=1,\dots,s} \{(f_i(\alpha_i, x) - \alpha_{i0})/t_i\} & \text{if } \exists i / f_i(\alpha_i, x) > \alpha_{i0} \text{ and } f_i(\alpha_i, x) \leqslant \alpha_{i0} + t_i,\ \forall i, \\ 0 & \text{otherwise}, \end{cases} \tag{12}$$

where $d = \max\{d_0, \min_{j=1,\dots,k} \{d_j\}\}$, for each constraint $t_i = \beta_i \cdot d > 0$ and clearly $\beta_i \|Dh(x)\|_q = \beta_i \max_i |d_i h_i(x)| \geqslant t_i$, $\forall i \in M$.

More explicitly, we get

$$\tilde{\vartheta}_C(x) = \sup\left\{\lambda \Big/ \sum_{j=1}^r \alpha_{ij} h_j(x) \leqslant \alpha_{i0} + t_i(1 - \lambda),\ x_j \geqslant 0,\ i = 1, \dots, s,\ \lambda \in (0, 1]\right\}.$$

The λ-cut of this modified $(g, 1)$-fuzzy feasible set is:

$$[\tilde{\vartheta}_C]_\lambda = \{x \in \mathbf{R}^r \mid \tilde{\vartheta}_C(x) \geqslant \lambda\} = \left\{x \in \mathbf{R}^r \;\middle|\; \sum_{j=1}^r \alpha_{ij} h_j(x) \leqslant \alpha_{i0} + t_i(1 - \lambda),\ x_j \geqslant 0,\ i = 1, \dots, s\right\}.$$

Let us fix λ and let x_λ denote the solution of the following problem:

$$\min\left\{\sum_{j=1}^{r} \gamma_j h_j(x) \mid x \in [\tilde{\vartheta}_C]_\lambda\right\},$$

where $y_\lambda = \sum_{j=1}^{r} \alpha_j h_j(x_\lambda) - (1-\lambda)\beta_0 d$, and evaluating this pair (x_λ, y_λ) in (5) we obtain

$$\tilde{\omega}(x_\lambda, y_\lambda) = g^{(-1)}(1-\lambda) = \lambda$$

and therefore

$$\tilde{\omega}^*(x_\lambda, y_\lambda) = \lambda.$$

Now, denote by $x(\lambda)$ the parametric solution of the following parametric mathematical programming problem $P(\lambda)$:

$$\min \sum_{j=1}^{r} \gamma_j h_j(x) \tag{13}$$

$$\text{s.t.} \sum_{j=1}^{r} \alpha_{ij} h_j(x) \leqslant \alpha_{i0} + t_i(1-\lambda), \quad i = 1, \ldots, s,$$

$$x_j \geqslant 0, \quad \lambda \in (0, 1].$$

As is well known, sufficient conditions for the existence of $x(\lambda)$ are that $[\tilde{\vartheta}_C]_\lambda$ be compact and the Lagrangian function of $P(\lambda)$ has a saddle point [8, 12]. Hence, the following results hold.

Theorem 1. *The optimal value in* (7) *using the linear function* $g(x) = 1 - x$ *is reached from the parametric programming problem* (13) *by*

$$x^* = x(\theta), \qquad y^* = y_\theta = \sum_{j=1}^{r} \gamma_j h_j(x^*) - (1-\theta)\beta_0 d \tag{14}$$

where $\theta = \sup_{\lambda \in (0,1]} \{\lambda \mid P(\lambda)$ *has optimal solution*$\}$; *moreover* $\vartheta^* = \theta$.

Proof. It is obvious that

$$\tilde{\vartheta}_C(x(\theta)) = \theta \geqslant \tilde{\vartheta}_C(x), \quad \forall x \in \mathbf{R}^r$$

and $\tilde{\omega}^*(x, y) = T_{g\infty}(\tilde{\omega}(x, y), \tilde{\vartheta}_C(x)) \leqslant \tilde{\vartheta}_C(x) \leqslant \theta$, $\forall x \in \mathbf{R}^r$, $\forall y \in \mathbf{R}$. Moreover $\tilde{\omega}(x(\theta), y_\theta) = \theta$, and $\tilde{\omega}^*(x(\theta), y_\theta) = \theta = \vartheta^* \geqslant \tilde{\omega}^*(x, y)$, $\forall x \in \mathbf{R}^r$, $\forall y \in \mathbf{R}$. On the other hand, if x^* is not optimal for $P(\theta)$ then there exists $y_\theta = y^* - (\sum_{j=1}^{r} \gamma_j h_j(x^*) - \sum_{j=1}^{r} \gamma_j h_j(x(\theta)))$ such that the pair $(x(\theta), y_\theta)$ satisfies (7). This contradicts the minimality of y^*. Therefore $x^* = x(\theta)$ and $y^* = y_\theta = \sum_{j=1}^{r} \gamma_j h_j(x^*) - (1-\theta)\beta_0 d$. □

Thus, the optimal solution for the approach presented in [8] has been obtained as a particular value of the parametric solution of (13).

When the generator function used, g', is not a linear function, and $[0, t]$, $t = g_0$, is the range of the function g', the following result is obtained:

Theorem 2. *Given the function* $g:[0, 1] \to [0, 1]$, $g(x) = 1 - x$, *and* g' *a generator function* $g':[0, 1] \to [0, g_0]$ *with* $g'(1) = 0$, $g'(0) = g_0$, *then there exists a strictly increasing function* $r:[0, 1] \to [0, t]$ *such that* $g' = t \circ g$.

Proof. Clearly we have the inverse function of g, $g^{-1}(y) = 1 - y$, which is a strictly decreasing function, and if we define the function $t = g' \circ g^{-1}$, then it satisfies the required conditions. □

If nonlinear functions g' are used, the fuzzy optimum set $\omega^*(x, y)$ is written as:

$$\omega^*(x, y) = (g')^{-1}(\Phi(x, y)/\|\overline{Dh}(x)\|_q), \tag{15}$$

$$\omega^*(x, y) = g^{-1}(t^{-1}(\Phi(x, y)/\|\overline{Dh}(x)\|_q)). \tag{16}$$

Since t^{-1} is a strictly increasing function, the optimum for (16) is reached in the solution obtained from Theorem 1 with $y^* = \sum_{i=1}^{r} \gamma_j h_j(x^*) - (1 - \vartheta_{g'}^*)\beta_0 d$.

Setting $\vartheta_g^* = w^*(x^*, y^*) = g^{-1}(\varphi(x^*, y^*))$, where $\varphi(x, y) = \Phi(x, y)/\|\overline{Dh}(x)\|_q$, then

$$\vartheta_{g'}^* = g'^{-1}(\varphi(x^*, y^*)) = g^{-1}(t^{-1}(\varphi(x^*, y^*))$$

so that $g(\vartheta_{g'}^*) = t^{-1}(\varphi(x^*, y^*)) = t^{-1}(g(\vartheta_g^*))$ and

$$\vartheta_{g'}^* = g^{-1}(t^{-1}(g(\vartheta_g^*))). \tag{17}$$

Thus, the above results imply that the optimal solution x^* for $\omega^*(x, y)$ is the same independently of the generator functions used. Moreover, by means of (17) we can obtain different degrees for the optimal solution (x^*, y^*) in the fuzzy optimum set $\omega^*(x, y)$, by using nonlinear generator functions. Clearly, if $\vartheta_g^* = 1$ then $\vartheta_{g'}^* = 1$ for all generator functions.

4. Numerical example

Consider the problem

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & -x_1 - 3x_2 \leqslant -9, \\ & 2x_1 + x_2 \leqslant 2, \\ & -4x_1 - 3x_2 \leqslant -17, \\ & x_j \geqslant 0, \end{aligned}$$

with $d_0 = 2$, $d_1 = 2$, $d_2 = 3$, $\beta_0 = 2$, $\beta_1 = 2$, $\beta_2 = 8$ and $\beta_3 = 3$ the tolerance margins for variables and fuzzy inequalities respectively

Then, since $d = \max\{d_0, \min_{j=1,\ldots,k}\{d_j\}\}$ and $t_i = \beta_i \cdot d$, one has $d = 2$, $t_1 = 4$, $t_2 = 16$ and $t_3 = 6$, and hence the associated parametric problem is

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & -x_1 - 3x_2 \leqslant -9 + 4(1 - \lambda), \\ & 2x_1 + x_2 \leqslant 2 + 16(1 - \lambda), \\ & -4x_1 - 3x_2 \leqslant -17 + 6(1 - \lambda), \\ & x_j \geqslant 0, \quad \lambda \in (0, 1]. \end{aligned}$$

Solving it, the parametric solution obtained is:

$$x(\lambda) = (2 + 0.666\lambda, 1 + 1.111\lambda), \quad \lambda \in (0, 0.704].$$

If we use different generator functions we obtain the results shown in Table 1.

Table 1. Solutions using different generator functions

$g(x)$	$t(z)$	ϑ^*	x^*	y^*
$1 - x$	z	0.704	(2.466, 1.78)	$4.246 - 4 \cdot 0.296 = 3.062$
$(1 - x)^p$	z^p	$1 - (0.296)^{1/p}$	(2.466, 1.78)	$4.246 - 4 \cdot (0.296)^{1/p}$
$(1 - x)^2$	z^2	0.456	(2.466, 1.78)	$4.246 - 4 \cdot 0.544 = 2.07$
$-\ln(x)$	$-\ln(1 - z)$	0.744	(2.466, 1.78)	$4.246 - 4 \cdot 0.256 = 3.222$

5. Conclusions

In this paper an alternative formulation of the optimality concept given by Kovács [7, 8] has been presented, which is based on parametric mathematical programming and involves the model presented in [8]. Together with this, a solution method has been provided, which allows us to obtain the solution associated to the optimal concept presented by Kovács [8] in an easier way. It has been shown that the optimal solution x^* is the same for every generator function, and the differences in the use of different generator functions are in the optimal objective function value and in the associated optimality degree.

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