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Hybrid crossover operators for real-coded genetic algorithms: an experimental study

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Abstract Most real-coded genetic algorithm research has focused on developing effective crossover operators, and as a result, many different types have been proposed. Some forms of crossover operators are more suitable to tackle certain problems than others, even at the different stages of the genetic process in the same problem. For this reason, techniques which combine multiple crossovers have been suggested as alternative schemes to the common practice of applying only one crossover model to all the elements in the population. Therefore, the study of the synergy produced by combining the different styles of the traversal of solution space associated with the different crossover operators is an important one. The aim is to investigate whether or not the combination of crossovers perform better than the best single crossover amongst them.

In this paper we have undertaken an extensive study in which we have examined the synergetic effects among real-parameter crossover operators with different search biases. This has been done by means of hybrid real-parameter crossover operators, which generate two offspring for every pair of parents, each one with a different crossover operator. Experimental results show that synergy is possible among real-parameter crossover operators, and in addition, that it is responsible for improving performance with respect to the use of a single crossover operator.

Keywords Real-coded genetic algorithms · Crossover operator · Hybrid crossover operators

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1 Introduction

In the initial formulation of *genetic algorithms* (GAs), the search space solutions are coded using the binary alphabet ([Gol89]); however, other coding types, such as real coding, have also been taken into account to deal with the representation of the problem. The real coding approach seems adequate when tackling optimisation problems of parameters with variables in continuous domains. The chromosome is a vector of floating point numbers, representing a solution of the problem. Obviously, both have the same length. GAs based on real-number representation are called real-coded GAs (RCGA s) ([Deb01a, Her98]).

The crossover operator is a method for sharing information between chromosomes. It has always been regarded as the main search operator in GAs ([DeJ92, Kit01]) because it exploits the available information in previous samples to influence future searches. This is why most RCGA research has been focused on developing effective real-parameter crossover operators, and as a result, many different possibilities have been proposed ([Deb01a, Her98]). In [Her03], a taxonomy is introduced to classify the crossover operators for RCGAs. It groups the models for this operator into different categories according to the way in which they generate the genes of the offspring from the genes of the parents. The empirical study of representative examples in all of the categories provides some clues as to the key features that have a positive influence on crossover behaviour.

Each crossover operator directs the search towards a different zone in the neighbourhood of the parents. The quality of the elements that belong to the visited region depends on the particular problem to be solved. This means that different crossover operators perform differently with respect to different problems, even at the different stages of the genetic process in the same problem. In fact, *no free lunch* theorems confirm this fact ([Wol97]). Thus, the simultaneous application of diverse crossover operators on the population could provide effective models that may be suited to many practical problems. In fact, some studies have been undertaken which examine the synergy produced by combining the different styles of the traversal of solution space associated with various crossover operators ([Dav89, Her00, Hon95, Hon98, Hon02, Spe95, Yoo02]). Their objective was to investigate whether or not a combination of crossovers perform better than the best single crossover amongst them.

It is to be expected that real-parameter crossover operators representing different groups of taxonomy suggested in [Her03] have dissimilar traversal styles of solution space. Thus, the study of the synergy derived from their combination becomes interesting, because it may reveal complementary properties that are required in order to build an effective coupling of real-parameter crossover operators. The aim of this paper is, in fact, to carry out an extensive study to examine the synergetic effects among real-parameter crossover operators that belong to different categories of this taxonomy. In order to do this we will design *hybrid real-parameter crossover operators*, which generate two offspring for every pair of parents, each one with a crossover operator that belongs to a different category. Hybrid operators represent a simple way to combine crossover operators, and thus, they constitute a manageable framework to analyse the synergetic effects of different real-parameter crossover operators.

The paper is set out as follows. In Sect. 2, we introduce relevant issues related to real-parameter crossover operators and outline the taxonomy for these operators proposed in [Her03]. In Sect. 3, we design hybrid real-parameter crossover operators, which combine two crossover operators included in different groups of the taxonomy. In Sect. 4, we describe the experimental study aimed at determining the goodness associated to the hybrid crossover operators (Sect. 4.2) and analyse the synergetic effects produced among their constituent crossover operators (Sect. 4.3). Finally, in Sect. 5, we point out some concluding remarks and summarise some possible future research areas related to this topic.

2 Real-parameter crossover operators

In this section, we deal with the main aspects of the crossover operators for RCGAs. In Sect. 2.1, we explain the three mechanisms involved in the application of the crossover operator. This is useful to understand the particular features of the crossover operators analysed in this paper. In Sect. 2.2, we examine the availability of these operators to propitiate different exploration or exploitation degrees. Finally, in Sect. 2.3, we introduce a taxonomy that groups real-parameter crossover operators into different categories depending on the way in which they generate the genes of the offspring from the genes of the parents. In Appendixes A and B, we include the description of the crossover operators used in this paper.

2.1 The crossover operator

The application of the crossover operator is carried out by means of three different mechanisms:

- *Mating selection mechanism* (MSM). This determines the way in which the chromosomes are mated for applying the crossover to them. The most common MSM pairs the parents randomly. However, other approaches have been proposed ([Fer01]).
- *Offspring generation mechanism* (OGM). This produces new chromosomes from each set of parents formed by the MSM. All the proposed OGMs for binary coding may be adapted to work with real coding. However, real coding offers the possibility of defining a wide variety of special OGMs that take advantage of its numerical nature. In general, the value of the genes corresponding to each position in the offspring are calculated by numerically combining the values of the genes of the parents in this position.
- *Offspring selection mechanism* (OSM). This mechanism chooses the offspring which will be population members out of all the offspring generated for each set of parents. One of the most widely used OSMs selects the best offspring to form the next population ([Her97, Wri91]).

Usually, the crossover operator is applied to pairs of chromosomes, generating two offspring for each one of them, which are introduced in the population ([Gol89]). However, *multi-parent* crossover operators have been proposed, which combine the features of more than two parents for generating the offspring ([Deb02, Kit99a, Ono97 Ort01, Som01, Tsu99]). Furthermore, crossover operators with *multiple descendants* have been presented ([Deb02, Esq97, Her97, Her02, Sat96, Wri91]), which produce more than two offspring for each group of parents. In this case, the OSM limits the number of offspring that will be population members. We should emphasise that in this paper we deal with crossover operators for real coding that require only two parents and generate only two offspring.

If we cross two parents, all the offspring may be created using the same OGM ([Deb02, Sat96]) or by means of different OGMs ([Her97]). In this paper, the former will be referred to as *homogeneous crossover operators* and the latter *hybrid crossover operators*.

2.2 Exploration and exploitation

Real-parameter crossover operators are able to produce exploration or exploitation (to different degrees) depending on the way in which they handle the current diversity of the population. They may either generate additional diversity starting from the current one (therefore exploration takes effect) or use this diversity for creating better elements (therefore exploitation comes into force). This is possible because of their self-adaptive features ([Bey01, Deb01b, Kit01]).

The performance of an RCGA on a particular problem will be strongly determined by the degrees of exploration and exploitation associated to the crossover operator being applied. In the following, we will introduce basic ideas about the availability of the crossover for adapting different exploration or exploitation degrees.

Let's consider $c_i^1, c_i^2 \in [a_i, b_i]$ two genes to be combined with $\alpha_i = \min\{c_i^1, c_i^2\}$ and $\beta_i = \max\{c_i^1, c_i^2\}$, the action interval, $[a_i, b_i]$, of these genes can be divided into three intervals: $[a_i, \alpha_i]$, $[\alpha_i, \beta_i]$, and $[\beta_i, b_i]$. These intervals bind three regions to which the resultant genes of some combination of the former may belong. In addition, considering a region $[\alpha_i', \beta_i']$ with $\alpha_i' \leq \alpha_i$ and $\beta_i' \geq \beta_i$ would seem reasonable. Fig. 1 shows this in graph form.

These intervals may be classified as exploration or exploitation zones as shown in Fig. 1. The interval with both genes being the extremes is an *exploitation* zone because any gene, g_i, generated by a crossover in this interval fulfils

$$\max\{|g_i - \alpha_i|, |g_i - \beta_i|\} \leq |\alpha_i - \beta_i|.$$

The two intervals that remain on both sides are *exploration* zones because the above property is not fulfilled. The region with extremes α_i' and β_i' could be considered as a *relaxed exploitation* zone. Therefore, exploration and/or exploitation degrees may be assigned to any crossover operator for RCGAs depending on the way in which these intervals are considered to generate genes.

The arithmetical crossover with $\lambda = 0.5$ (Appendix A) is a clear example of the exploitative crossover operator. On the other hand, this operator will show exploration for $\lambda > 1$ or $\lambda < 0$. An example of crossover showing relaxed exploitation is BLX-α (Appendix A). Nomura et al. ([Nom01]) provide a formalisation of this operator to analyse the relationship between the chromosome probability density functions before and after its application, assuming an infinite population. They state that BLX-α spreads the distribution of the chromosomes when $\alpha > (\sqrt{3} - 1)/2$ or otherwise reduces it. This property was verified through simulations. In particular, the authors observed that BLX-0.0 makes the variances of the distribution of the chromosomes decrease, reducing the distribution, whereas BLX-0.5 makes the variances of the distribution increase, spreading the distribution.

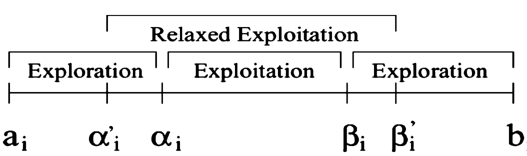

Fig. 1 Action interval for c_i^1 and c_i^2

2.3 Taxonomy

In [Her03], a taxonomy is presented which classifies the crossover operators for RCGAs (those applied only on two parents) into different groups, focusing on the features associated with the OGMs that are applied to the parents to obtain the offspring. This includes whether they preserve the genes of the parents in the offspring, whether the genes of the offspring are obtained from an aggregation function in which its arguments are the genes of the parents, or whether the genes in the offspring are generated from a probability distribution defined in the neighbourhoods of the genes of the parents. The taxonomy includes the following groups:

- *Discrete Crossover Operators* (DCOs). This category groups all the crossover operators proposed for binary coding, which are directly applicable to real coding. It includes the two-point and uniform crossover operators (Appendix A). With these crossovers, the value of each gene in the offspring coincides with the value of this gene in one of the parents ($h_i \in \{c_i^1, c_i^2\}$), i.e., the values of the genes in the parents are not transformed numerically to obtain the values of the genes in the offspring. Geometrically, DCOs generate a corner of the hypercube defined by the component of the two parents. The effect of these operators, according to the intervals of the generation of genes, is shown in Fig. 2.
- *Aggregation Based Crossover Operators* (ABCOs). These include operators that use an aggregation function that numerically combines the values of the genes of the parents to generate the value of the genes of the offspring. If $[a_i, b_i]$ is the action interval for the i-th gene, an aggregation function, $f_i : [a_i, b_i] \to [a_i', b_i']$ $([a_i', b_i'] \subseteq [a_i, b_i])$ should be provided. Then, the value for the i-th gene of the offspring is calculated as $f_i(c_i^1, c_i^2)$. The arithmetical and geometrical crossover operators (Appendix A) are examples of ABCOs. In the case of the arithmetical crossover, the aggregation function is a linear combination of c_i^1 and c_i^2. Graphically, ABCOs act as shown in Fig. 3.

As seen in Figure 3, the ABCOs may generate genes in the exploitation interval or in the exploration interval.

- *Neighbourhood-Based Crossover Operators* (NBCOs). This group includes crossovers that determine the genes of the offspring extracting values from intervals defined in neighbourhoods associated with the genes of the parents throughout probability distributions. Examples of NBCOs are

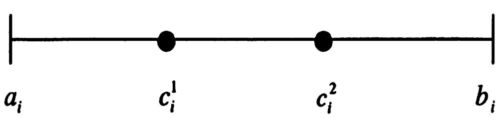

Fig. 2 Effects of the DCOs

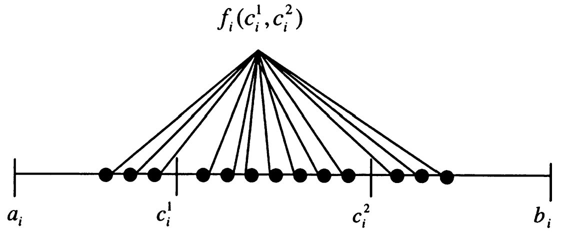

Fig. 3 Possible gene values calculated by ABCOs from c_i^1 and c_i^2

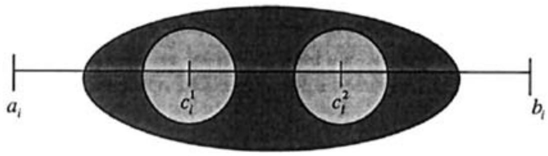

Fig. 4 Neighbourhoods taken into account by NBCOs

BLX-α, simulated binary crossover, and fuzzy recombination (Appendix A), which are based on uniform, polynomial, and triangular probability distributions, respectively. Fig. 4 represents graphically the neighbourhoods considered by NBCOs

The main difference between ABCOs and NBCOs is that ABCOs are *deterministic* crossovers, i.e., given two parents, the resultant offspring shall always be the same, whereas NBCOs include a random component, i.e., they are non deterministic.

Table 1 contains the real-parameter crossover operators used in this paper, along with the category to which they belong. As we have mentioned, their formulation may be found in Appendix A. In addition, Appendix B is devoted to the description of the dynamic heuristic crossover operators. These operators have been selected from the ones studied in [Her03] according to their good performance.

In the following, we make some comments about the exploration and exploitation features of the crossover operators in Table 1:

- At gene level, two-point and uniform crossovers (DCOs) do not offer diversity, because they maintain the genes of the parents in the offspring. In this sense, we can say that they show exploitative features (Fig. 1). However, at chromosome level, they offer diversity, because new chromosomes (the offspring) are created concatenating different segments of genes of the parents. Thus, from this point of view, they are explorative operators.
- For the experiments (Sect. 4), we have chosen values for the parameters associated to the arithmetical and geometrical crossovers that make these operators show exploitation. In addition, the dynamic heuristic operator used (Appendix B) has a strong tendency towards exploitation ([Her96a]).
- All NBCOs included in Table 1 generate genes in relaxed exploitation intervals. The diversity degrees offered by them will depend on the value chosen for their associated parameters (α, η, and d).

Table 1 Real-parameter crossover operators

Crossover operator	Taxonomy group
Two-point (2P) Uniform(U)	DCO
Arithmetical (A) Geometric (G) Dynamic Heuristic (DH)	ABCO
BLX-α (α = 0.3 and α = 0.5) Simulated Binary (SBX-η) (η = 2 and η = 5) Fuzzy Recombination (FR-d) (d = 0.5)	NBCO

3 Hybrid real-parameter crossover operators

The idea behind crossover is that by combining features from two good parents crossover will often produce even better offspring ([Esh95]). However, the efficiency of crossover for genetic search is governed by the relationship between the crossover biases (its traversal style of solution space) and the search problem itself. A particular crossover operator becomes more effective when its search bias is adjusted to the structure of the problem to be solved. Thus, some forms of crossover operators are more suitable for solving certain problems than others, even at the different stages of the genetic process in the same problem. The *no free lunch* theorems confirm this fact ([Wol97]).

An interesting idea to devise crossover-based techniques, which may be suited to most practical problems, would consist of the simultaneous application of diverse crossover operators on the population. In fact, some studies have been conducted in which the synergy produced by combining the different styles of the traversal of solution space associated to various crossover operators has been examined ([Dav89, Her00, Hon95, Hon98, Hon02, Spe95, Yoo02]). Their objective was to investigate whether or not a combination of crossovers perform better than the best single crossover amongst them.

3.1 Crossover Combination Techniques

There have been different attempts to find synergetic crossover operators:

1. *Hybrid crossover operators*. These crossovers use different kinds of crossover operators to produce diverse offspring from the same parents. For example, in [Her97], an hybrid real-parameter crossover operator is presented, which generates four offspring for each pair of parents, applying two explorative crossovers and two exploitative crossovers to them. The two most promising offspring of the four substitute their parents in the population.

2. *Heterogeneous distributed* GAs. In [Her00], a distributed RCGA model maintains, in parallel, several sub-populations that are processed by independent GAs that apply different forms of crossover operators. These operators are differentiated according to their associated exploration and exploitation properties and the degree thereof. Other distributed GA models that make distinctions between the sub-populations by applying GAs with different crossovers are found in [Sch94] and [Eib98]. In this case, each sub-population competes with other sub-populations, in such a way that it gains or loses individuals depending on its evolution quality in relation to the others.
3. *Adaptive crossover operator probabilities*. A set of crossover operators is available, each with a probability of being used. For each reproduction event, a single operator is selected probabilistically according to the set of operator probabilities. In addition, an adaptive process dynamically adjusts the operator probabilities during the course of evolving a solution. For example, in [Dav89] and [Jul95], those operators that create and cause the generation of better chromosomes are given higher probabilities, i.e., they should be used more frequently. On the other hand, operators producing offspring with a fitness that is lower than that of the parents should be used less frequently. Another approach for adaptation involves self-adaptation ([Spe95, Tus98]); operator probabilities are directly coded onto each member of the population and this allows them to evolve, i.e., they undergo mutation and recombination.

Two models of adaptive real-parameter crossover operator probabilities are found in [Ono99] and [Her96b]. In [Ono99], two complementary crossover operators are considered, UNDX ([Ono97]) and uniform crossover ([Sys89]). UNDX may efficiently optimise functions with strong epistasis among parameters. However, it has difficulties in exploring the search space. Uniform crossover has contrary properties (suitable search space exploration and deficient exploitation on functions with epistasis among parameters), and therefore, it complements the UNDX operation. A mechanism was introduced for adapting the operator probabilities according to the characteristics of a given function. In [Her96b], an RCGA applies two different crossover operators; one with exploitation properties and another with exploration properties. An operator probability parameter defines the frequency of the application of the exploitative operator. Every five generations, a fuzzy logic controller evaluates two population diversity measures to adjust this parameter.

3.2 Proposals of hybrid real-parameter crossover operators

The examples of crossover operators which were grouped into three taxonomy categories (Sect. 2.3) are expected to provide different traversal styles of search space. Thus, in order to carry out a study of the synergy among operators of different categories, we present several examples of *hybrid real-parameter crossover operators*. They generate two offspring for every pair of parents by applying two crossover operators, each one selected from a different category. Hybrid crossovers are a simple way of combining crossover operators and, therefore, constitute a framework which facilitates the study of the synergetic effects of different real-parameter crossover operators.

We have chosen a number of representative crossover examples of the three taxonomy categories (those in Table 1) and have built three kinds of hybrid crossover operators, ABCO-NBCO, ABCO-DCO, and DCO-NBCO, as shown in Tables 2, 3, and 4.

These three types of hybridisation allow us to analyse the effects derived from the union between different exploration and exploitation characteristics:

- The ABCO-NBCO hybridisation merges the relaxed exploitation supplied by NBCOs with the strong exploitative inclination of ABCOs. The relaxed exploitation intervals considered by BLX-α with α= 0.5 and SBX-η with η=2 include an important proportion of exploration intervals, favouring the production of high population diversity levels. In this case, their union with ABCOs seems to be very reasonable. In fact, crossover combination strategies have been proposed that incorporate examples of

Table 2 ABCO-NBCO hybrid real-parameter crossover operators

NBCO	ABCO		
	A	G	DH
BLX-0.3	A&BLX-0.3	G&BLX-0.3	DH&BLX-0.3
BLX-0.5	A&BLX-0.5	G&BLX-0.5	DH&BLX-0.5
SBX-2	A&SBX-2	G&SBX-2	DH&SBX-2
SBX-5	A&SBX-5	G&SBX-5	DH&SBX-5
FR-0.5	A&FR-0.5	G&FR-0.5	DH&FR-0.5

Table 3 ABCO-DCO hybrid real-parameter crossover operators

ABCO	DCO	
	U	2P
A	A&U	A&2P
G	G&U	G&2P
DH	DH&U	

Table 4 DCO-NBCO hybrid real-parameter crossover operators

NBCO	DCO	
	U	2P
BLX-0.3	U&BLX-0.3	2P&BLX-0.3
BLX-0.5	U&BLX-0.5	2P&BLX-0.5
SBX-2	U&SBX-2	2P&SBX-2
SBX-5	U&SBX-5	2P&SBX-5
FR-0.5	U&FR-0.5	2P&FR-0.5

explorative and exploitative crossover operators ([Her96b, Her97, Ono99, Wri91]).

- The idea behind DCO-NBCO operators is that of applying operators that propitiate diversity at gene level (NBCOs) and operators that introduce diversity at chromosome level (DCOs) at the same time.
- Finally, ABCO-DCO operators attempt to extract the maximum effectiveness from the parents by using two kinds of crossover operators with exploitation properties at gene level. In addition, the diversity induced by DCOs at chromosome level may be useful to complement the powerful exploitation of ABCOs.

4 Experiments

Minimisation experiments on the test suite described in Appendix C were carried out in order to study the behaviour of the hybrid crossover operators presented in the previous section. In Sect. 4.1, we describe the algorithms built in order to do this. In Sect. 4.2, we show the results and draw some conclusions. In Sect. 4.3, we examine the synergetic effects among the crossovers that form the best hybrid crossovers. The basis for this judgement is simple: does the hybridisation of crossovers perform better than the best single crossover amongst them? Finally, in Sect. 4.4, we deal with some aspects related to the hybrid crossover as the best performing operator obtained in these experiments.

4.1 Algorithms

In our experiments we have taken a *generational* RCGA model that applies the *non-uniform* mutation operator ([Mic92]). This operator has been widely used with good results ([Her98]). The selection probability calculation follows *linear ranking* ([Bak85]) ($\eta_{\min} = 0.75$) and the sampling algorithm is the *stochastic universal sampling* ([Bak87]). The *elitist strategy* ([DeJ75]) is also considered. This involves making sure that the best performing chromosome always survives intact from one generation to the next.

We have implemented two different types of RCGA examples:

- The examples in the first group use homogeneous crossover operators; two offspring are produced for every pair of parents using the same crossover. The operators in Table 1 have been used to implement these RCGAs. We have taken $\lambda = 0.25$ and $\omega = 0.25$ for the arithmetical and geometrical crossover operators, respectively. The homogeneous dynamic heuristic crossover generates one offspring with the dynamic-dominated crossover and the other with the dynamic-biased crossover (Appendix B).
- The examples in the second group apply the hybrid crossover operators in Table 2. All the crossover operators in Table 1 produce two offspring (Appendix A). In order to design hybrid operators, each crossover operator compounding the hybridisation returns only one offspring, which is chosen at random. In the case of using the arithmetical or geometrical crossover operators, the offspring is built with $\lambda = 0.5$ and $\omega = 0.5$, respectively. The version of dynamic heuristic crossover involved in the hybrid crossovers is the DBD one (Sect. B.2 in Appendix B).

The RCGAs will be denoted in the same way as their corresponding crossover operator. The population size is 61 individuals, the probability of updating a chromosome by mutation is 0.125, and the crossover probability is 0.6. We carried out all the algorithms 30 times, each one with a maximum of 100,000 evaluations.

4.2 Analysis of the results

The results obtained are shown in Tables D1–D7 in Appendix D. The performance measures used are the following:

- *A* performance: average of the best-fitness function found at the end of each run.
- *SD* performance: standard deviation.
- *B* performance: best of the fitness values averaged as *A* performance.

Moreover, a *t*-test (at 0.05 level of significance) was applied in order to ascertain if differences in the *A* performance for the best crossover operator are significant when compared with the one for the other crossovers in the respective table. The *T* column in these tables shows the result of the *t*-test. In this column, the crossover with the best *A* performance value is marked with **, and the direction of any significant differences is denoted either by a plus sign (+) for an improvement in *A* performance or an approximate sign (≅) for non-significant differences.

Table 5 summarises the results of Tables D1–D7. It shows the percentages in which each crossover operator has obtained the best *A* performance on all test functions. Its columns have the following information:

- *Total best*. Percentage of test functions in which the crossover operator achieves the best *A* performance, without considering the *t*-test.
- *Total best/Similar t-test*. Percentage of test functions in which the crossover operator obtains either the best *A* behaviour or one similar to the best, according to the t-test. The two groups of RCGA examples are ordered based on this performance measure.

Taking into consideration these results, we would make the following comments:

- According to the *"Total best/Similar t-test"* performance measure, the two best algorithms are 2P&SBX-2 and DH&BLX-0.5, which are based on

Table 5 Results for the real-parameter crossover operators

Homogeneous crossovers	Total best	Total best/ similar *t*-test
DH	30.76%	46.14%
FR-0.5	15.38%	38.45%
SBX-2	0%	38.45%
SBX-5	0%	30.76%
BLX-0.3	0%	30.76%
BLX-0.5	0%	23.07%
A	0%	23.07%
U	0%	15.38%
2P	0%	7.69%
G	0%	7.69%
Hybrid crossovers		
2P&SBX-2	7.69%	53.83%
DH&BLX-0.5	7.69%	53.83%
DH&SBX-2	7.69%	46.14%
DH&FR-0.5	0%	46.14%
DH&U	0%	46.14%
DH&BLX-0.3	0%	38.45%
2P&BLX-0.5	0%	38.45%
2P&BLX-0.3	0%	38.45%
DH&SBX-5	0%	30.76%
U&BLX-0.5	0%	30.76%
DH&2P	7.69%	30.76%
U&SBX-2	7.69%	30.76%
DH&A	7.69%	23.07%
A&BLX-0.5	7.69%	23.07%
U&BLX-0.3	0%	23.07%
2P&SBX-5	0%	23.07%
2P&FR-0.5	0%	23.07%
A&SBX-2	0%	23.07%
U&FR-0.5	0%	23.07%
U&SBX-5	0%	15.38%
A&BLX-0.3	0%	15.38%
U&A	0%	7.69%
U&G	0%	7.69%
2P&G	0%	7.69%
A&SBX-5	0%	7.69%

hybrid crossover operators. This means that the hybridisation of different real-parameter crossover operators is a recommended strategy to improve the effectiveness of this operator.

- The hybrid operators which combine the dynamic heuristic crossover and crossovers that belong to the NBCO group (DH&BLX-0.5, DH&SBX-2, and DH&FR-0.5) have provided promising results. In fact, DH&BLX-0.5 achieves one of the best results in Table 5. Furthermore, the homogenous dynamic heuristic crossover (DH) and homogenous NBCO crossovers (FR-0.5, SBX-2, SBX-5, and BLX-0.3) show satisfactory behaviour (which agrees with the results obtained in [Her03]). We may conclude that, in general, the combination of promising real-parameter crossover operators allows effective hybrid operators to be obtained.
- Although DCO crossover operators perform inadequately working alone, their behaviour is enhanced when combined with NBCOs. This can be seen most clearly in the case of the two-point crossover, which when combined with SBLX-2 (2P&SBX-2) becomes one of the best hybrid crossover operators in Table 5.

Table 6 Study of synergy in DH&BLX-0.5

	Total best	Total best/ similar *t*-test
* DH	76.9%	84.59%
BLX-0.5	23.07%	38.45%
* DH&BLX-0.5	61.52%	76.9%
DH	38.45%	61.52%

Table 7 Study of synergy in 2P&SBX-2

	Total best	Total best/ similar *t*-test
2P	30.76%	69.21%
* SBX-2	69.21%	69.21%
* 2P&SBX-2	76.9%	92.28%
SBX-2	23.07%	46.14%

Table 8 Study of synergy in DH&SBX-2

	Total best	Total best/ similar *t*-test
* DH	69.21%	69.21%
SBX-2	30.76%	53.83%
DH&SBX-2	46.14%	53.83%
* DH	53.83%	61.52%

Table 9 Study of synergy in DH&FR-0.5

	Total best	Total best/ similar *t*-test
DH	46.14%	69.21%
* FR-0.5	53.83%	69.21%
* DH&FR-0.5	61.52%	84.59%
FR-0.5	38.45%	61.52%

4.3 Study of the synergy

This section provides an extensive empirical study of the synergy amongst the constituent operators of the best five hybrid crossover operators in Table 5 (2P&SBX-2, DH&BLX-0.5, DH&SBX-2, DH&FR-0.5, and DH&U). For these hybrid operators, Tables 6–10 include two types of comparisons; firstly, between the two constituent crossovers, and secondly, among the best one and the hybrid operator (the '*' symbol identifies the best algorithm). In this way, we tackle synergy as suggested in [Yoo02]:

"Consider two crossovers CX1 and CX2 and assume without loss of generality that crossover CX1 performs better than CX2 when used alone. If the mixing of CX1 and CX2 performs better than the sole usage of CX1, we say that crossovers CX1 and CX2 have synergy."

The percentages in these tables are obtained taking into account only the results in Tables 14–20 of the two algorithms that have been compared. In addition, we

Table 10 Study of synergy in DH&U

	Total best	Total best/ similar *t*-test
* DH	84.59%	100%
U	15.38%	30.76%
DH&U	38.45%	76.9%
* DH	61.52%	76.9%

have assumed that algorithm *A* is better than *B* when *A* achieves better "*Total best/similar t-test*" performance measure than *B*. In the case of a tie, *A* becomes the best when it obtains better "*Total best*" performance measure than *B*.

This study reveals that synergy is produced amongst the operators that compound DH&BLX-0.5, 2P&SBX-2, and DH&FR-0.5. This is a crucial result in the case of DH&BLX-0.5 and 2P&SBX-2 (which were the best crossover operators in Table 5), because it suggests that the synergy caused by combining different real-parameter crossover operators allows hybrid crossovers to have a positive influence on RCGA performance. Thus, hybrid crossovers become attractive, because they may bring together properties that are needed in an effective crossover operator, which is difficult to achieve from the use of a single crossover operator.

Another important point is that synergy has been possible amongst operators from different categories of the taxonomy. In particular, DH&BLX-0.5 and DH&FR-0.5 are representative of the combination ABCO-NBCO, whereas 2P&SBX-2 is an example of the combination of DCO and NBCO. It is interesting to note that although a poor performance is achieved from the use of the single two-point crossover (see Table 5), its synergy with SBX-2 allows the hybrid 2P&SBX-2 to be one of the most competitive crossover operators which has been studied in this paper.

4.4 DH&BLX-0.5 vs. 2P&SBX-2

Table 11 has been introduced in order to compare the two hybrid crossover operators that have given the best results in the experiments, DH&BLX-0.5 and 2P&SBX-2 (see Table 5).

We clearly observe that DH&BLX-0.5 outperforms 2P&SBX-2. DH&BLX-0.5 combines two crossover operators with complementary properties:

- BLX-α with $\alpha = 0.5$ is an operator which favours high exploration as explained in Section 2.2.
- DH provides diversity levels that decrease with the passage of time, introducing a heuristic local tuning (exploitation) that becomes effective for RCGA performance ([Her96a]).

Table 11 2P&SBX-2 vs. DH&BLX-0.5

	Total best	Total best/similar *t*-test
2P&SBX-2	23.07%	53.83%
DH&BLX-0.5	76.9%	84.59%

The suitable relationship between the exploration associated to BLX-α and the exploitation caused by DH induces synergetic effects that allow DH&BLX-0.5 to consistently outperform all the crossover operators studied in this paper.

Table 12 compares DH&BLX-0.5 and DH&BLX-0.3 (which combines DH with BLX-α with a lower value for α, α=0.3), which aims to analyse the importance of the exploration provided by BLX-0.5 on the behaviour of DH&BLX-0.5.
This table shows that DH&BLX-0.3 loses performance with regard to DH&BLX-0.5. This means that the high diversity provided by BLX-0.5 is essential to achieve the robust behaviour exhibited by DH&BLX-0.5.

As $0.5 > (\sqrt{3} - 1)/2$, BLX-0.5 is able to spread the distribution of the chromosomes, which is not possible with α=0.3, because $0.3 < (\sqrt{3} - 1)/2$ (see Sect. 2.2). In this way, BLX-0.5 supplies the adequate exploration capabilities to complement the exploitation features associated with DH, making adequate synergy among them possible.

We have introduced Table 13 with the aim of confirming this property. For every test problem, it outlines the *percentage of success* of DH (defined as the percentage of times the offspring produced by DH is better than the one generated by BLX-0.5, from the total of crossover events produced throughout the run).

A visual inspection of this table allows one to remark the following conclusions:

- The effectiveness of DH (see Table 5) may explain the generalized high values for the percentage of success of this operator. Its exploitative capability

Table 12 DH&BLX-0.5 vs. DH&BLX-0.3

	Total best	Total best/ similar *t*-test
DH&BLX-0.5	53.83%	92.28%
DH&BLX-0.3	46.14%	61.52%

Table 13 Percentage of success of DH

Test Problem	% DH
Sphere	82.41%
Schwefel 1.2	73.32%
Rastrigin	76.89%
Griewangk	77.98%
Expansion of F10	77.95%
Polynomial fitting problem	74.89%
Frequency modulation sound	75.88%
Systems of linear equations	75.13%
Rosenbrock	74.49%
Ackley	81.90%
Bohachevsky	81.08%
Watson	3.47%
Coville	76.49%

favours the creation of good offspring, as compared with the offspring returned by BLX-0.5, which are properly destined to the promotion of diversity in the population.

- There are differences between the percentages of success of DH for the different problems. Easy problems, such as the Sphere function, present the highest values, whereas the complex problems, such as the Schwefel's function, the Polynomial fitting problem, and the Watson's function, have associated the lowest values. In this case, the exploration of BLX-0.5 allows the localization of promising solutions for problems with search spaces that exhibit many difficulties.
- The percentage of success of DH for the Watson's function is very low (3.47%). This problem is highly complex, and uniquely the action of BLX-0.5 allows the RCGA to advance towards better regions of the search space.

These results show how hybrid crossovers may adapt to problems with different challenges, thanks to the incorporation of two crossover operators with different styles of the traversal of solution space.

5 Conclusions

This paper presented a model of hybrid crossover operators as a suitable tool to facilitate the study of the synergy amongst real-parameter crossover operators with different search biases. They generate two offspring using two crossovers chosen from different groups of the taxonomy proposed in [Her03]. The main conclusions derived from the results of the experiments carried out are as follows:

- Hybrid crossover operators achieve a finer performance than homogeneous crossover operators. Thus, the hybridisation of real-parameter crossover operators shows promise as a strategy to improve the effectiveness of this genetic operator.
- Synergy is possible amongst operators from different categories of taxonomy. In particular, some combinations of neighbourhood-based crossover operators and crossovers that belong to the other categories (ABCOs and DCOs) have exhibited this feature.
- The joint application of BLX-α (α=0.5) and dynamic heuristic crossovers induces an appropriate relationship between exploration and exploitation to produce profitable synergetic effects, allowing a robust operation to be achieved for test functions with different characteristics.

Moreover, we may point out that the application hybrid crossover operators does not represent an significant increase on computational complexity, with regards to the common practice of applying only one crossover.

In conclusion, we can say that hybrid crossovers are very promising and indeed worth further study. We are currently continuing our research of hybrid crossover operators based on multi-parent crossover operators and crossover operators with multiple descendants (Sect. 2). In addition, we intend to examine the synergy involved in other forms of crossover combinations, such as heterogeneous distributed GAs and mechanisms for the adaptive crossover operator probabilities (Sect. 4). Finally, another future research area concerns the study of the influence of the selective pressure of the selection mechanism on the performance of hybrid crossover operators. This is an important aspect because the amount of exploration performed by crossover is severely affected by the degree of selective pressure of the selection mechanism ([Esh89, Her00]). Recent studies on the selective pressure provided by different selection mechanisms will be useful to carry out such research. ([Mot02]).

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References

Ackley DH (1987) A connectionist machine for genetic hillclimbing. Kluwer Boston

Baker JE (1985) Adaptative selection methods for genetic algorithms. In: Grefenstette JJ (ed) Proceedings of the first international conference on genetic algorithms and their applications, L. Erlbraum Associates, Hillsdale, MA, pp 101—111

Baker JE (1987) Reducing bias and inefficiency in the selection algorithm. In: Grefenstette JJ, (ed) Proceedings of the second international conference on genetic algorithms and their applications, Lawrence Erlbaum, Hillsdale, NJ, pp 14–21

Beyer H-G, Deb K (2001) On self-adaptive features in real-parameter evolutionary algorithms. IEEE Trans. Evol. Comput. 5(3): pp 250–270

Bremermann HJ, Rogson M, Salaff S (1966) Global properties of evolution processes. In: Natural automata and useful simulations, Pattee HH, Edlsack EA, Frin L, Callahan AB (eds) Spartan, Washington, DC, pp 3–41

Davis L (1989) Adapting operator probabilities in genetic algorithms. In: Proceedings of the third international conference on genetic algorithms, David Schaffer J (ed) Morgan-Kaufmann Publishers, San Mateo, pp 61–69

Deb K, Agrawal RB (1995) Simulated binary crossover for continuous search space. Complex Syst. 9: 115–148

Deb K (2001) Multi-objective optimization using evolutionary algorithms. Wiley, Chichester

Deb K, Beyer H (2001) Self-adaptive genetic algorithms with simulated binary crossover. Evol. Comput. J. 9(2): 195–219

Deb K, Anand A, Joshi D (2002) A computationally efficient evolutionary algorithm for real-parameter evolution. Evol. Comput. J. 10(4): 371–395

De Jong KA (1975) An analysis of the behavior of a class of genetic adaptive systems. Doctoral Dissertation, University of Michigan

De Jong KA, Spears WM (1992) A formal analysis of the role of multi-point crossover in genetic algorithms. Ann. Math. Artif. Intell. 5(1): 1–26

Eiben AE, Sprinkhuizen-Kuyper IG, Thijssen BA (1998) Competing crossovers in an adaptive GA framework. In: Proceedings IEEE conference evolutionary computation, IEEE Press, Piscataway, NJ, pp 787–792

Eshelman LJ, Caruana A, Schaffer JD (1989) Biases in the crossover landscape. In: Proceedings of the third international con-

ference on genetic algorithms, David Schaffer J (ed) Morgan Kaufmann Publishers, San Mateo, CA, pp 86–91
Eshelman LJ, Schaffer JD (1993) Real-coded genetic algorithms and interval-schemata. In: Foundations of genetic algorithms 2, Whitley LD (ed) Morgan Kaufmann Publishers, San Mateo, CA, pp 187–202
Eshelman LJ, Schaffer JD (1995) Productive recombination and propagating and preserving schemata. In: Foundations of genetic algorithms 3, Whitley LD, Vose MD (eds) (Morgan Kaufmann Publishers, San Mateo, CA, pp 299–313
Eshelman LJ, Mathias KE, Schaffer JD (1997) Convergence controlled variation. In: Foundations of genetic algorithms 4, Belew R, Vose M (eds) Morgan Kaufmann Publishers, San Mateo, CA, pp 203–224
Esquivel S, Leiva A, Gallard R (1997) Multiple crossover per couple in genetic algorithms. In: Proceedings of the 4th IEEE international conference on evolutionary computation (ICEC'97), IEEE Press, Piscataway, NJ, pp 103–106
Fernandes C, Rosa A (2001) A study on non-random mating and varying population size in genetic algorithms Using a royal road function. In: Proceedings of the 2001 congress on evolutionary computation, IEEE Press, Piscataway, NJ, pp 60–66
Goldberg DE (1989) Genetic algorithms in search, optimization, and machine learning. Addison-Wesley, Reading, MA
Griewangk AO (1981) Generalized descent of global optimization. J. Optimization Theory Appl. 34: 11–39
Herrera F, Lozano M, Verdegay JL (1996) Dynamic and heuristic fuzzy connectives based crossover operators for controlling the diversity and convergence of real-coded genetic algorithms. Int. J. Intell. Syst. 11: 1013–1041
Herrera F, Lozano M (1996) Adaptation of genetic algorithm parameters based on fuzzy logic controllers. In: Genetic Algorithms and Soft Computing, Herrera F, Verdegay JL (eds) Physica-Verlag, pp 95–125
Herrera F, Lozano M, Verdegay JL (1997) Fuzzy connectives based crossover operators to model genetic algorithms population diversity. Fuzzy Set Syst 92(1): 21–30
Herrera F, Lozano M, Verdegay JL (1998) Tackling real-coded genetic algorithms: Operators and tools for behavioural analysis. Artif. Intell. Rev. 12(4): 265–319
Herrera F, Lozano M (2000) Gradual distributed real-coded genetic algorithms. IEEE Trans. Evol. Comput. 4(1): 43–63
Herrera F, Lozano M, Pérez E, Sánchez AM, Villar P (2002) Multiple crossover per couple with selection of the two best offspring: An experimental study with the BLX-a crossover operator for real-coded genetic algorithms. In: Garijo FJ, Riquelme JC, Toro M (eds) IBERAMIA Lecture notes in artificial intelligence 2527, Springer, Berlin, pp 392–401
Herrera F, Lozano M, Sánchez AM (2003) A taxonomy for the crossover operator for real-coded genetic algorithms. An experimental study. Int. J. Intell. Syst. 18(3): 309–338
Hong I, Kahng AB, Moon BR (1995) Exploiting synergies of multiple crossovers: initial studies. In: Proceedings of the second IEEE conference on evolutionary computation, IEEE Press, Piscataway, NJ, pp 245–250
Hong T-P, Wang H-S (1998) Automatically adjusting crossover ratios of multiple crossover operators. J. Inform. Sci. Eng. 14(2): 369–390
Hong T-P, Wang H-S, Lin W-Y, Lee W-Y (2002) Evolution of appropriate crossover and mutation operators in a genetic process. Appl. Intell. 16: 7–17
Julstrom BA (1995) What have you done for me lately? adapting operator probabilities in a steady-state genetic algorithm. In: Proceedings of the sixth international conference on genetic algorithms, Eshelman L. (ed) Morgan Kaufmann Publishers, San Francisco, pp 81–87
Kita H, Kobayashi S (1999) Multi-parental extension of the unimodal normal distribution crossover for real-coded genetic algorithms. In: Proceedings of the international conference on evolutionary computation'99, IEEE Press, Piscataway, NJ, pp 646–651
Kita H (2001) A comparison study of self-adaptation in evolution strategies and real-coded genetic algorithms. Evol. Comput. J. 9(2): 223–241
Michalewicz Z (1992) Genetic algorithms + data structures = evolutionary programs. Springer, Berlin
Michalewicz Z, Nazhiyath G, Michalewicz M (1996) A note on usefulness of geometrical crossover for numerical optimization problems. In: Proceedings of the 5th annual conference on evolutionary programming, MIT Press, Cambridge, pp 305–312
Motoki T (2002) Calculating the expected loss of diversity of selection schemes. Evol. Comput. J. 10(4): 397–422
Nomura T, Shimohara K (2001) An analysis of two-parent recombinations for real-valued chromosomes in an infinite population. Evol. Comput. J. 9(3): 283–308
Ono I, Kobayashi S (1997) A real-coded genetic algorithm for function optimization using unimodal normal distribution crossover. In: Proceedings of the seventh international conference on genetic algorithms, Bäck T (ed) Morgan Kaufmmann, San Mateo, CA, pp 246–253
Ono I, Kita H, Kobayashi S (1999) A robust real-coded genetic algorithm using unimodal normal distribution crossover augmented by uniform crossover: effects of self-adaptation of crossover probabilities. In: Proceedings of the genetic and evolutionary computation conference (GECCO-1999), Morgan Kaufmann, San Mateo, CA, pp 496–503
Ortiz D, Hervas C, Muñoz J (2001) Genetic algorithm with crossover based on confidence intervals as an alternative to least squares estimation for nonlinear models. In: Fourth metaheuristics international conference, pp 343–347
Reynolds RC, Chung C (1997) Knowledge-based self-adaptation in evolutionary programming using cultural algorithms. In: Proceedings of 1997 international conference on evolutionary computation, IEEE Press, Piscataway, NJ, pp 71–76
Satoh H, Yamamura M, Kobayashi S (1996) Minimal generation gap model for gas considering both exploration and explotation. In: Proceedings methodologies for the conception, design and application of intelligent systems (IIZUKA'96), pp 494–497
Schwefel H-P (1981) Numerical optimization of computer models. Wiley, Chichester
Schlierkamp-Voosen D, Mühlenbein H (1994) Strategy adaptation by competing subpopulations. In: Parallel problem solving from nature 3, Davidor Y, Schwefel H-P, Männer R (eds) Springer, Berlin, Germany, pp 199–208
Someya H, Yamamura M (2001) Genetic algorithm with search area adaptation for the function optimization and its experimental analysis. In: Proceedings of the 2001 congress on evolutionary computation, IEEE Press, Piscataway, NJ, pp 933–940
Spears WM (1995) Adapting crossover in a genetic algorithm. In Proceedings of the fourth annual conference on evolutionary programming, McDonnell JR, Reynolds RG, Fogel D (eds) The MIT Press, pp 367–384
Storn R, Price K (1995) Differential evolution – a simple and efficient adaptive scheme for global optimization over continuous spaces. Technical report TR-95–012, International computer science institute, Berkeley, CA
Syswerda G (1989) Uniform crossover in genetic algorithms. In: Proceedings of the third international conference on genetic algorithm, Schaffer JD (ed) Morgan Kaufmann Publishers, San Mateo, CA, pp 2–9
Törn A, Antanas Z (1989) Global optimization. Lecture notes in computer science 350, Springer, Berlin
Tsutsui S, Fujimoto Y (1993) Forking genetic algorithm with blocking and shrinking modes. In: Proceedings of the fifth international conference on genetic algorithms, Forrest S (ed) Morgan Kaufmann Publishers, San Mateo San Mateo, CA, pp 206–213
Tsutsui S, Yamamura M, Higuchi T (1999) Multi-parent recombination with simplex crossover in real-coded genetic algorithms. In: Proceedings of the genetic and evolutionary computation conference (GECCO-99), Morgan Kaufmann, San Mateo, CA, pp 657–664

Tuson AL, Ross P (1998) Adapting operator settings in genetic algorithms. Evol. Comput. 6(2): 161–184

Voigt HM, Mühlenbein H, Cvetkovic D (1995) Fuzzy recombination for the breeder genetic algorithm. In: Proceedings of the sixth international conference on genetic algorithms, Eshelman L (ed) Morgan Kaufmann Publishers, San Mateo, CA, pp 104–111

Whitley D, Beveridge R, Graves C, Mathias K (1995) Test driving three 1995 genetic algorithms: new test functions and geometric matching. J. Heuristics 1: 77–104

Wolpert DH, Macready WG (1997) No free lunch theorems for optimization. IEEE Trans. Evol. Comput. 1(1): 67–82

Wright A (1991) Genetic algorithms for real parameter optimization. In: Rawlin GEJ (ed) Foundations of Genetic Algorithms 1, Morgan Kaufmann, San Mateo, CA, pp 205–218

Yoon H-S, Moon B-R (2002) An Empirical study on the synergy of multiple crossover operators. IEEE Trans. Evol. Comput. 6(2): 212–223

Appendix A. Crossover operator for real coding

Let us assume that $C_1 = (c_1^1, ..., c_n^1)$ and $C_2 = (c_1^2, ..., c_n^2)$ are two chromosomes that have been selected to apply the crossover operator to them. Below, we describe the operation of the crossover operators for RCGAs considered in this paper, and show their effects in graph form.

- *Two-point crossover* ([Esh89]). Two points of crossover are randomly selected $(i, j \in \{1, 2, ..., n-1\}$ with $i < j$, and the segments of the parent, defined by them, are exchanged for generating two offspring:

$$H_1 = (c_1^1, c_2^1, \ldots, c_i^2, c_{i+1}^2, \ldots, c_j^2, c_{j+1}^1, ..., c_n^1),$$

$$H_2 = (c_1^2, c_2^2, \ldots, c_i^1, c_{i+1}^1, ..., c_j^1, c_{j+1}^2, \ldots, c_n^2).$$

- *Uniform crossover* ([Sys89]). Two offspring are created, $H_k = (h_1^k, ..., h_i^k, ..., h_n^k)$, $k = 1, 2$. The value of each gene in the offspring is determined by the random uniform choice of the values of this gene in the parents:

$$h_i^k = \begin{cases} c_i^1 & if\; u = 0, \\ c_i^2 & if\; u = 1, \end{cases}$$

  u being a random number which can have a value of zero or one.

- *Arithmetical crossover* ([Mic92]). Two offspring are produced, $H_k = (h_1^k, ..., h_i^k, ..., h_n^k)$, $k = 1, 2$, where

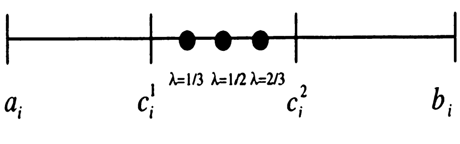

Fig. 5 Arithmetical crossover with different values forλ

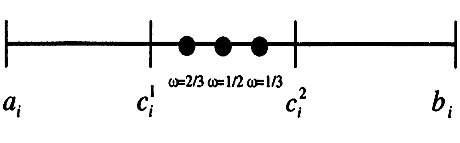

Fig. 6 Geometrical crossover with different values for ω

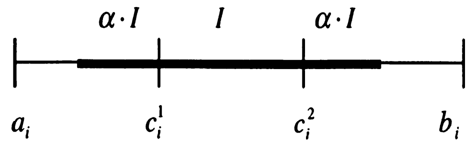

Fig. 7 BLX-α

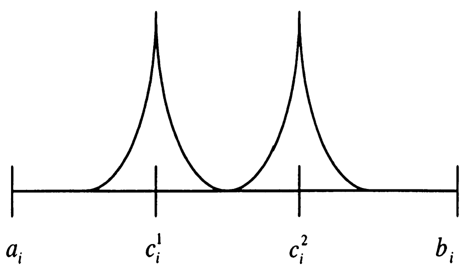

Fig. 8 Simulated binary crossover

  $h_i^1 = \lambda \cdot c_i^1 + (1-\lambda) \cdot c_i^2$ and $h_i^2 = \lambda \cdot c_i^2 + (1-\lambda) \cdot c_i^1$, here$\lambda \in [0, 1]$.

- *Geometrical crossover* ([Mic96]). Two offspring are built, $H_k = (h_1^k, \ldots, h_i^k, \ldots, h_n^k)$, $k = 1, 2$, where $h_i^1 = c_i^{1^\omega} \cdot c_i^{2^{1-\omega}}$and $h_i^2 = c_i^{2^\omega} \cdot c_i^{1^{1-\omega}}$, with $\omega \in [0, 1]$.
- *BLX*-α ([Bre66, Esh93]). Two offspring are generated, $H_k = (h_1^k, \ldots, h_i^k, \dot{s}, h_n^k)$, $k = 1, 2$, where h_i^k is a randomly (uniformly) chosen number from the interval $[C_{\min} - I\alpha, C_{\max} + I\alpha]$, where

$$C_{\max} = \max\{c_i^1, c_i^2\},$$

$$C_{\min} = \min\{c_i^1, c_i^2\}, \text{and}$$

$$I = C_{\max} - C_{\min}.$$

- *Simulated binary crossover* ([Deb95, Deb01a]): Two offspring are generated, $H_k = (h_1^k, \ldots, h_i^k, \ldots, h_n^k)$, $k = 1, 2$, where:

$$h_i^1 = \frac{1}{2} \cdot \left[(1-\beta_k) \cdot c_i^1 + (1+\beta_k) \cdot c_i^2\right] \text{and}$$

$$h_i^2 = \frac{1}{2} \cdot \left[(1+\beta_k) \cdot c_i^1 + (1-\beta_k) \cdot c_i^2\right],$$

  $\beta_k (\geq 0)$ is a sample from a random number generator having the density:

$$p(\beta) = \begin{cases} \frac{1}{2} \cdot (\eta+1)\beta^\eta, & if\; 0 \leq \beta \leq 1 \\ \frac{1}{2} \cdot (\eta+1)\frac{1}{\beta^{\eta+2}}, & if\; \beta > 1. \end{cases}.$$

  This distribution can easily be obtained from a uniform $u(0, 1)$ random number source by the transformation:

$$\beta(u) = \begin{cases} (2u)^{\frac{1}{\eta+1}} & if\; u(0,1) \leq \frac{1}{2} \\ [2(1-u)]^{-\frac{1}{\eta+1}} & if\; u(0,1) > \frac{1}{2}. \end{cases}$$

- *Fuzzy recombination* ([Voi95]). Two offspring are produced, $H_k = (h_1^k, \ldots, h_i^k, \ldots, h_n^k)$, $k = 1, 2$. The probability that the i-th gene in an offspring has the value v_i is given by the distribution $p(v_i) \in \{\phi(c_i^1), \phi(c_i^2)\}$ where $\phi(c_i^1)$ and $\phi(c_i^2)$ are tri-

Prob. Dist.	Min.	Modal	Max.
$\phi(c_i^1)$	$c_i^1 - d \cdot I$	c_i^1	$c_i^1 + d \cdot I$
$\phi(c_i^2)$	$c_i^2 - d \cdot I$	c_i^2	$c_i^2 + d \cdot I$

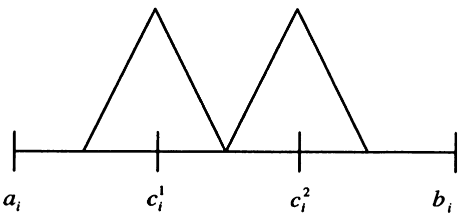

Fig. 9 Fuzzy recombination

angular probability distributions having the following features ($c_i^1 \le c_i^2$ is assumed and $I = |c_i^1 - c_i^2|$):

Appendix B. Dynamic heuristic crossover operators

The *dynamic heuristic* crossovers ([Her96a]) merge the features of two types of crossover operators:

- *Heuristic crossovers* ([Her96a, Wri91, Sch94]. They generate offspring close to the best parent, with the objective of leading the search process towards the most promising zones.
- *Dynamic crossovers* ([Her96a]). They keep a suitable sequence between the exploration and the exploitation along the GA run: "*to protect the exploration in the initial stages and the exploitation later*".

Dynamic heuristic crossover operators allow the level of heuristic effect to be dependent on the current generation in which they are applied. At the beginning, this level is low and diversity is high (offspring are distant from parents), later on, the heuristic effects gradually increase.

In order to describe these operators, two steps are followed: in Sect. B.1, we define function families for the combination of genes and in Sect. B.2, we use these families to design dynamic heuristic crossover operators.

B.1 Function families for the combination of genes

With regards to the exploration and exploitation intervals shown in Fig. 1, in [Her97], three monotone and non-decreasing functions are proposed: F, S, and M, defined from $[a,b] \times [a,b]$ into $[a,b]$, $a, b \in \Re$, which fulfil:

$$\forall c, c' \in [a,b],\ F(c,c') \le \min\{c,c'\},$$
$$S(c,c') \ge \max\{c,c'\}, \text{ and}$$
$$\min\{c,c'\} \le M(c,c') \le \max\{c,c'\}.$$

Each one of these functions allows us to combine two genes giving results belonging to each one of the aforementioned intervals. Therefore, each function will have different exploration or exploitation properties depending on the range being covered by it.

For a RCGA with a maximum number of generations $g_{\max}$, in [Her96a], three families of functions were proposed: a family of F functions, $\mathrm{F} = (F^1, \ldots, F^{g_{\max}})$, a family of S functions, $\mathrm{S} = (S^1, \dot{s}, S^{g_{\max}})$, and a family of M functions, $\mathrm{M} = (M^1, \ldots, M^{g_{\max}})$, which for $1 \le t \le g_{\max} - 1$ fulfil:

$$\forall c, c' \in [a,b] \quad F^t(c,c') \le F^{t+1}(c,c') \text{ and}$$
$$F^{g_{\max}}(c,c') \approx \min\{c,c'\},$$

$$\forall c, c' \in [a,b] \quad S^t(c,c')^{t+1}(c,c') \text{ and}$$
$$S^{g_{\max}}(c,c') \approx \max\{c,c'\},$$

$$\forall c, c' \in [a,b] \quad M^t(c,c') \ge M^{t+1}(c,c') \text{ or}$$
$$M^t(c,c') \le M^{t+1}(c,c')\ \forall t \text{ and}$$
$$M^{g_{\max}}(c,c') \approx M_{\lim}(c,c'),$$

where $M_{\lim}$ is an M function called M limit function. We shall denote M^+ or M^- an M function family fulfilling the first and the second part of the last property, respectively.

F and S function families may be built using a parameterised t-norm T^q converging to the minimum and a parameterised t-conorm G^q converging to the maximum, respectively. In this study, we use the Dubois parameterised t-norm and t-conorm:

$$x, y \in [0,1], 0 \le q \le 1, \quad T^q(x,y) = \frac{x \cdot y}{x \vee y \vee q}$$

and

$$G^q(x,y) = 1 - \frac{(1-x) \cdot (1-y)}{(1-x) \vee (1-y) \vee q}.$$

To do this, a transformation of the genes to be combined is needed, from the interval $[a,b]$ into $[0,1]$, and then, the result into $[a,b]$. Furthermore, we need a function $\partial(\cdot)$ to transform the values of $\{1, \ldots, g_{\max}\}$ into the range of q. All this may be summed up in the following equations. Given the Dubois parameterised t-norm T^q and t-conorm G^q, we build two function families $\mathrm{F} = (F^1, \ldots, F^{g_{\max}})$ and $\mathrm{S} = (S^1, \ldots, S^{g_{\max}})$as:

$$\forall c, c' \in [a,b] \ \ 1 \le t \le g_{\max}\ F^t(c,c')$$
$$= a + (b-a) \cdot T^{\delta_F(t)}(s,s'),$$

$$\forall c, c' \in [a,b] \ \ 1 \le t \le g_{\max}\ S^t(c,c')$$
$$= a + (b-a) \cdot G^{\delta_S(t)}(s,s'),$$

where $s = \frac{c-a}{b-a}$, $s' = \frac{c'-a}{b-a}$, and

$$\delta_F(t) = \delta_S(t) = \frac{1}{\sqrt{t}}$$

are the transformation functions.

We may obtain M function families using a parameterised averaging function. An example of these functions is:

$$\forall x, y \in [0,1], P^q(x,y) = \sqrt[q]{\frac{x^q + y^q}{2}}, -\infty \le q \le \infty.$$

This may be achieved as follows:

$$\forall c, c' \in [a,b],\ 1 \le t \le g_{\max}\ M^t(c,c') = a + (b-a) \cdot P^{\delta_M(t)}(s,s').$$

In particular, we could obtain an M^+ and an M^- function family using

$$\delta_M = \delta_{M^+} = 1 + \ln\left(\frac{g_{\max}}{t}\right) \text{ and}$$

$$\delta_M = \delta_{M^-} = 1 + \ln\left(\frac{t}{g_{\max}}\right), \text{ respectively.}$$

B.2 Dynamic heuristic crossovers

Let's suppose $C_1^t = (c_1^{1t}, \ldots, c_n^{1t})$ and $C_2^t = (c_1^{2t}, \ldots, c_n^{2t})$, two chromosomes which were selected to apply the crossover to them in a generation t. Let's also suppose that C_1^t is the one with the best fitness. Then, we may generate $H^t = (h_1^t, \ldots, h_n^t)$ using one of the following dynamic heuristic crossovers ([Her96a]):

- *Dynamic-dominated crossover*

$$h_i^t = \begin{cases} F^t(c_i^{1t}, c_i^{2t}) & if\ c_i^{1t} \le c_i^{2t} \\ S^t(c_i^{1t}, c_i^{2t}) & otherwise \end{cases} \qquad i = 1, \ldots n,$$

where F^t and S^t belong to an F and an S function family, respectively. We may use the Dubois F and S families (Sect. A.2) to obtain the Dubois dynamic dominated crossover operators.

Dynamic dominated crossovers have heuristic exploration properties, which allow useful diversity to be introduced into the RCGA population.

- *Dynamic- biased crossover*

$h_i^t = M_i^t(c_i^{1t}, c_i^{2t})$, $i = 1, \ldots n$, where M_i^t belongs to a function family M^+ if $c_i^{1t} \le c_i^{2t}$ or to a function family M^-, otherwise. Both have the same M limit function, which fulfils:

$$\left|c_i^{1t} - M_{\lim}(c_i^{1t}, c_i^{2t})\right| \le \left|c_i^{2t} - M_{\lim}(c_i^{1t}, c_i^{2t})\right|.$$

The following parameterised averaging operator was used to build such an operator:

$$\forall x, y \in [0,1], 0 \le q \le 1, \quad P^q(x,y) = q \cdot x + (1-q) \cdot y$$

Also, we consider the following initial conditions:

[1]. $M^1(c_i^1, c_i^2) = \frac{c_i^1 + c_i^2}{2}$,

[2]. $M_{\lim}(c_i^1, c_i^2) = (1-\lambda) \cdot c_i^1 + \lambda \cdot c_i^2$.

where λ is calculated as follows:

$$\lambda = 1 - \frac{f(C_1)}{f(C_1) + f(C_2)},$$

$f(\cdot)$ being the fitness function. Dynamic-biased crossover shows heuristic exploitation properties, which induces a biased convergence towards the best elements.

- In this paper, we have defined a third version of a dynamic heuristic operator. It produces an offspring, $H^t = (h_1^t, \ldots, h_n^t)$, applying, randomly, the dynamic-biased or dynamic-dominated crossover operator to generate each gene of H^t. This operator is denoted as DHBD. Fig. 10 shows the operation of this operator.

Appendix C. Test suite

Sphere model ([DeJ75]).

$$f_{Sph}(x) = \sum_{i=1}^{n} x_i^2$$

$$-5.12 \le x_i \le 5.12,\ n = 25,\ f_{Sph}(x^*) = 0.$$

Schwefel's function 1.2 ([Sch81]).

$$f_{Sch}(x) = \sum_{i=1}^{n} \sum_{j=1}^{i} x_j^2$$

$$-65.536 \le x_i \le 65.536,\ n = 25,\ f_{Sch}(x^*) = 0.$$

Generalised Rastrigin's function ([Tör89]).

$$f_{Ras}(x) = a \cdot n + \sum_{i=1}^{n} x_i^2 - a \cdot \cos(\omega \cdot x_i)$$

$$a = 10,\ \omega = 2 \cdot \pi,\ -5.12 \le x_i \le 5.12,\ n = 25,\ f_{Ras}(x^*) = 0.$$

Griewangk's function ([Gri81]).

$$f_{Gri}(x) = \frac{1}{d} \sum_{i=1}^{n} x_i^2 - \Pi_{i=1}^{n} \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$$

$$d = 4000,\ -600 \le x_i \le 600,\ n = 25,\ f_{Gri}(x^*) = 0.$$

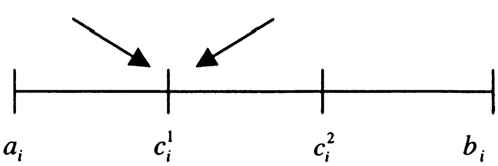

Fig. 10 Dynamic heuristic crossover

Expansion of F10 ([Whi95]).

$$e_{F10}(x) = f_{10}(x_1, x_2) + \cdots f_{10}(x_{i-1}, x_i) \cdots + f(x_n, x_1)$$
$$f_{10}(x, y) = (x^2 + y^2)^{0.25} \cdot \left[\sin^2(50 \cdot (x^2 + y^2)^{0.1}) + 1\right]$$
$$x, y \in (-100, 100],\ e_{F10}(x^*) = 0.$$

Generalised Rosenbrock's function ([DeJ75]).

$$f_{Ros}(x) = \sum_{i=1}^{n-1} (100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2)$$
$$-5.12 \leq x_i \leq 5.12,\ n = 25,\ f_{Ros}(x^*) = 0.$$

Systems of linear equations ([Esh97]).

The problem to be solved is to obtain the elements of a vector X, given the matrix A and vector B in the expression: $A \cdot \mathrm{X} = \mathrm{B}$. The evaluation function used for these experiments is:

$$P_{sle}(x_1, ..., x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} \cdot x_j) - b_j.$$

Clearly, the best value for this objective function is $P_{sle}(x^*) = 0$. Inter-parameter linkage (i.e., non-linearity) is controlled easily in systems of linear equations; their non-linearity does not deteriorate as increasing numbers of parameters are used, and they have proven to be quite difficult.

We have studied an example of a ten-parameter problem. Its matrices are as follows:

$$\begin{vmatrix} 5 & 4 & 5 & 2 & 9 & 5 & 4 & 2 & 3 & 1 \\ 9 & 7 & 1 & 1 & 7 & 2 & 2 & 6 & 6 & 9 \\ 3 & 1 & 8 & 6 & 9 & 7 & 4 & 2 & 1 & 6 \\ 8 & 3 & 7 & 3 & 7 & 5 & 3 & 9 & 9 & 5 \\ 9 & 5 & 1 & 6 & 3 & 4 & 2 & 3 & 3 & 9 \\ 1 & 2 & 3 & 1 & 7 & 6 & 6 & 3 & 3 & 3 \\ 1 & 5 & 7 & 8 & 1 & 4 & 7 & 8 & 4 & 8 \\ 9 & 3 & 8 & 6 & 3 & 4 & 7 & 1 & 8 & 1 \\ 8 & 2 & 8 & 5 & 3 & 8 & 7 & 2 & 7 & 5 \\ 2 & 1 & 2 & 2 & 9 & 8 & 7 & 4 & 4 & 1 \end{vmatrix} \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 40 \\ 50 \\ 47 \\ 59 \\ 45 \\ 35 \\ 53 \\ 50 \\ 55 \\ 40 \end{vmatrix}$$

Frequency modulation sound parameter identification ([Tsu93]).

The problem is to specify six parameters a_1, w_1, a_2, w_2, a_3, w_3 of the frequency modulation sound model represented by

$$y(t) = a_1 \cdot \sin(w_1 \cdot t \cdot \theta + a_2 \cdot \sin(w_2 \cdot t \cdot \theta + a_3 \cdot \sin(w_3 \cdot t \cdot \theta))),$$

with $\theta = (2 \cdot \pi/100)$. The fitness function is defined as the sum of square errors between the evolved data and the model data as follows:

$$P_{fms}(a_1, w_1, a_2, w_2, a_3, w_3) = \sum_{t=0}^{100} (y(t) - y_0(t))^2,$$

where the model data are given by the following equation:

$$y_0(t) = 1.0 \cdot \sin(5.0 \cdot t \cdot \theta + 1.5 \cdot \sin(4.8 \cdot t \cdot \theta + 2.0 \cdot \sin(4.9 \cdot t \cdot \theta))).$$

Each parameter is in the range –6.4–6.35. This is a highly complex multi-modal problem, having strong epistasis, with minimum value $P_{fms}(x^*) = 0$.

Polynomial fitting problem ([Sto95]).

This problem consists of finding the coefficients of the following polynomial in z:

$$P(z) = \sum_{j=0}^{2k} c_j \times z^j,\ k > 0$$

is an integer such that $P(z) \in [-1, 1]$ for $z \in [-1, 1]$, and $P(1.2) \geq T_{2k}(1.2)$and $P(-1.2) \geq T_{2k}(-1.2)$, where $T_{2k}(z)$ is a Chebychev polynomial of degree 2 k.

The solution to the polynomial fitting problem consists of the coefficients of $T_{2k}(z)$. This polynomial oscillates between –1 and 1 when its argument z is between -1 and 1. Outside this range the polynomial rises steeply in direction of high positive ordinate values. This problem has its roots in electronic filter design and challenges an optimisation procedure by forcing it to find parameter values with grossly different magnitudes, something very common in technical systems. The Chebychev polynomial employed here is:

$$T_8(z) = 1 - 32 \cdot z^2 + 160 \cdot z^4 - 256 \cdot z^6 + 128 \cdot z^8.$$

It is a nine-parameter problem. The pseudo-code algorithm shown below was used in order to transform the constraints of this problem into an objective function to be minimised, called P_{Chev}. We consider that $C = (c_0, \ldots, c_8)$ is the solution to be evaluated and $P_C(z) = \sum_{j=0}^{8} c_j \times z^j$.

Choose $p_0, \ldots, p_{100}$ *from* $[-1, 1]$;
$R = 0$;
For $i = 0, \ldots, 100$ *do*
  If $(-1 > P_C(p_i)$ *or* $P_C(p_i) > 1)$ *then*
    $R \leftarrow R + (1 - P_C(p_i))^2$;
  If $(P_C(1.2) - T_8(1.2) < 0)$ *then*
    $R \leftarrow R + (P_C(1.2) - T_8(1.2))^2$;
  If $(P_C(-1.2) - T_8(-1.2) < 0)$ *then*
    $R \leftarrow R + (P_C(-1.2) - T_8(-1.2))^2$;
Re*turn* R;

Each parameter (coefficient) is in the range −512–512. The objective function value of the optimum is $P_{Chev}(C^*) = 0$.

Ackley's function ([Ack87]).

$$f_{Ack}(x) = -a \cdot \exp\left(-b \cdot \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}\right) - \exp\left(\frac{1}{n}\sum_{i=1}^{n} \cos \omega \cdot x_i\right) + a + e$$

$$a = 20,\ b = 0.2,\ \omega = 2 \cdot \pi,\ -32.768 \le x_i \le 32.768,\ n = 25,\ f_{Ack}(x^*) = 0.$$

Bohachevsky's function ([Rey97]).

$$f_{Boh}(x) = x_1^2 + 2x_1^2 - 0.3\cos(3\pi x_1)\cos(4\pi x_2) + 0.3$$
$$-6 \le x_i \le 6,\ f_{Boh}(x^*) = 0.$$

Watson's function ([Rey97]).

$$f_{Wat}(x) = \sum_{i=1}^{30}\left(\sum_{j=1}^{5}(j a_i^{j-1} x_{j+1}) - \left[\sum_{j=1}^{6} a_i^{j-1} x_j\right]^2 - 1\right)^2 + x_1^2$$

$$a_i = \frac{i-1}{29},\ -2 \le x_i \le 2,\ f_{Wat}(x^*) = 2.288000e-3.$$

Colville's function ([Rey97]).

$$f_{Col}(x) = 100(x_1^2 - x_2)^2 + (1 - x_1)^2 + 90(x_3^2 - x_4) + (1 - x_3)^2 + 0.1((1 - x_2)^2 + (1 - x_4)) + 19.8(x_2 - 1)(x_4 - 1)$$
$$-10 \le x_i \le 10,\ f_{Col}(x^*) = 0.$$

Appendix D. Results of the Experiments

Table 14 Results for the sphere model and Schwefel's function 1.2

Sphere	B	A	T	SD	Schwefel	B	A	T	SD
A	1.74E-09	1.09E-08	+	6.53E-09	A	1.15E+01	4.03E+01	+	2.68E+01
G	2.09E-09	1.35E-08	+	1.21E-08	G	1.30E+02	5.94E+02	+	5.10E+02
BLX-0.3	1.27E-11	7.51E-11	+	5.35E-11	BLX-0.3	8.04E+00	3.37E+01	+	1.56E+01
BLX-0.5	6.12E-07	6.31E-06	+	8.11E-06	BLX-0.5	5.76E+02	1.36E+03	+	2.60E+02
SBX-2	4.38E-10	1.97E-09	+	1.17E-09	SBX-2	7.09E-01	7.56E+00		4.28E+00
SBX-5	6.00E-11	2.76E-10	+	2.08E-10	SBX-5	1.14E+01	9.54E+01	+	7.97E+01
FR-0.5	4.62E-12	1.30E-11	+	6.52E-12	FR-0.5	1.10E+00	8.97E+00		7.08E+00
2P	8.82E-10	3.77E-09	+	3.61E-09	2P	1.24E+02	4.78E+02	+	2.56E+02
U	1.73E-09	1.06E-08	+	8.13E-09	U	2.64E+02	7.21E+02	+	3.44E+02
DH	1.35E-15	1.37E-1f4	**	9.63E-15	DH	1.91E+01	6.04E+01	+	2.99E+01
A&U	8.87E-10	3.04E-09	+	1.38E-09	A&U	1.07E+01	4.28E+01	+	2.36E+01
G&U	8.36E-10	4.50E-09	+	4.30E-09	G&U	6.66E+01	1.74E+02	+	7.51E+01
U&BLX-0.5	3.30E-12	1.54E-11	+	1.14E-11	U&BLX-0.5	1.30E+00	8.41E+00		5.99E+00
U&BLX-0.3	6.76E-12	4.38E-11	+	3.07E-11	U&BLX-0.3	2.89E+01	8.03E+01	+	4.46E+01
U&SBX-5	7.14E-12	4.67E-11	+	5.09E-11	U&SBX-5	3.28E+01	1.51E+02	+	7.33E+01
U&SBX-2	9.84E-12	1.15E-10		2.41E-10	U&SBX-2	3.35E+00	1.28E+01	+	8.87E+00
U&FR-0.5	6.91E-13	4.51E-12	+	5.09E-12	U&FR-0.5	7.66E+00	3.04E+01	+	2.09E+01
A&2P	7.40E-10	3.61E-09	+	2.87E-09	A&2P	1.27E+01	3.54E+01	+	1.76E+01
G&2P	9.78E-10	4.13E-09	+	3.65E-09	G&2P	6.72E+01	2.28E+02	+	1.13E+02
2P&BLX-0.5	3.39E-12	2.07E-11	+	1.93E-11	2P&BLX-0.5	5.63E-01	4.90E+00		3.55E+00
2P&BLX-0.3	3.31E-12	3.65E-11	+	2.32E-11	2P&BLX-0.3	1.56E+01	5.49E+01	+	2.40E+01
2P&SBX-5	2.47E-11	1.02E-10	+	9.69E-11	2P&SBX-5	1.82E+01	1.94E+02	+	1.47E+02
2P&SBX-2	2.71E-11	2.00E-10	+	1.88E-10	2P&SBX-2	6.06E-02	4.55E+00	**	4.45E+00
2P&FR-0.5	1.12E-12	1.01E-11	+	1.35E-11	2P&FR-0.5	4.45E+00	2.69E+01	+	2.03E+01
A&BLX-0.5	2.66E-10	1.00E-09	+	8.88E-10	A&BLX-0.5	4.86E+00	1.46E+01	+	6.02E+00
A&BLX-0.3	1.01E-09	3.56E-09	+	2.08E-09	A&BLX-0.3	8.03E+00	2.32E+01	+	1.17E+01
A&SBX-5	3.03E-10	1.56E-09	+	1.07E-09	A&SBX-5	7.52E+00	3.19E+01	+	1.81E+01
A&SBX-2	3.04E-10	1.10E-09	+	7.03E-10	A&SBX-2	3.00E+00	1.16E+01	+	7.59E+00
DH&BLX-0.5	6.44E-14	3.34E-13	+	2.29E-13	DH&BLX-0.5	2.12E+00	1.34E+01	+	7.53E+00
DH&BLX-0.3	1.05E-07	2.53E-07	+	9.27E-08	DH&BLX-0.3	1.95E+00	1.18E+01	+	5.88E+00
DH&SBX-5	3.64E-09	1.41E-08	+	7.21E-09	DH&SBX-5	6.31E+00	2.30E+01	+	1.25E+01
DH&SBX-2	2.07E-08	7.87E-08	+	4.67E-08	DH&SBX-2	8.66E-01	7.91E+00		1.03E+01
DH&FR-0.5	6.32E-14	3.29E-13	+	1.89E-13	DH&FR-0.5	4.72E+00	2.50E+01	+	1.59E+01
DH&2P	5.10E-14	1.49E-12	+	1.33E-12	DH&2P	2.29E+01	9.46E+01	+	5.07E+01
DH&U	1.68E-13	1.40E-12	+	1.45E-12	DH&U	2.75E+01	7.86E+01	+	3.62E+01

Table 15 Results for Rastrigin's and Griewank's functions

Rastrigin	B	A	T	SD	Griewangk	B	A	T	SD
A	9.94E-01	3.97E+00	+	2.59E+00	A	3.59E-06	1.78E-02		1.87E-02
G	1.39E+01	1.94E+01	+	3.80E+00	G	7.42E-06	1.38E-02		1.15E-02
BLX-0.3	4.97E+00	7.86E+00	+	1.80E+00	BLX-0.3	1.10E-08	1.54E-02		1.56E-02
BLX-0.5	6.08E+01	8.72E+01	+	1.25E+01	BLX-0.5	5.06E-03	5.29E-01	+	2.16E-01
SBX-2	6.96E+00	1.36E+01	+	4.56E+00	SBX-2	3.22E-07	1.91E-02		2.28E-02
SBX-5	2.98E+00	7.13E+00	+	2.15E+00	SBX-5	8.69E-08	2.32E-02	+	2.51E-02
FR-0.5	1.19E+01	1.96E+01	+	4.84E+00	FR-0.5	3.21E-09	7.71E-03	**	9.60E-03
2P	1.52E-07	6.96E-01	+	7.77E-01	2P	7.07E-07	2.65E-02	+	2.45E-02
U	6.03E-07	6.96E-01	+	7.33E-01	U	2.15E-06	2.22E-02	+	1.93E-02
DH	8.52E-13	1.13E-11	**	1.09E-11	DH	1.76E-12	9.67E-03		1.32E-02
A&U	1.09E-06	4.44E+00	+	2.33E+00	A&U	7.18E-07	2.09E-02	+	1.65E-02
G&U	6.96E+00	1.42E+01	+	3.23E+00	G&U	1.16E-06	1.39E-02		1.46E-02
U&BLX-0.5	5.96E+00	1.66E+01	+	5.07E+00	U&BLX-0.5	1.89E-09	1.31E-02		1.16E-02
U&BLX-0.3	2.98E+00	6.46E+00	+	1.67E+00	U&BLX-0.3	7.20E-09	2.50E-02	+	2.07E-02
U&SBX-5	9.94E-01	4.21E+00	+	1.57E+00	U&SBX-5	1.17E-08	2.31E-02	+	2.46E-02
U&SBX-2	4.97E+00	1.01E+01	+	2.70E+00	U&SBX-2	1.24E-08	1.85E-02	+	1.53E-02
U&FR-0.5	4.97E+00	1.01E+01	+	2.93E+00	U&FR-0.5	1.05E-09	1.18E-02		1.10E-02
A&2P	9.94E-01	3.91E+00	+	2.00E+00	A&2P	6.47E-07	2.08E-02	+	1.72E-02
G&2P	4.97E+00	1.37E+01	+	3.91E+00	G&2P	1.14E-06	1.83E-02		1.89E-02
2P&BLX-0.5	7.95E+00	1.73E+01	+	4.12E+00	2P&BLX-0.5	1.74E-09	1.72E-02		2.01E-02
2P&BLX-0.3	1.98E+00	6.83E+00	+	2.14E+00	2P&BLX-0.3	1.09E-08	1.92E-02		2.01E-02
2P&SBX-5	9.94E-01	4.31E+00	+	1.71E+00	2P&SBX-5	1.75E-08	1.41E-02		1.32E-02
2P&SBX-2	3.53E-08	7.72E+00	+	2.73E+00	2P&SBX-2	4.66E-08	2.17E-02		2.51E-02
2P&FR-0.5	5.96E+00	1.02E+01	+	2.70E+00	2P&FR-0.5	7.88E-10	2.44E-02	+	1.94E-02
A&BLX-0.5	2.12E+00	5.96E+00	+	1.70E+00	A&BLX-0.5	2.34E-07	1.47E-02		1.62E-02
A&BLX-0.3	1.46E-01	4.62E+00	+	2.24E+00	A&BLX-0.3	1.40E-06	2.19E-02	+	1.99E-02
A&SBX-5	9.94E-01	6.40E+00	+	2.48E+00	A&SBX-5	4.94E-07	2.19E-02	+	1.71E-02
A&SBX-2	5.48E-07	6.57E+00	+	2.40E+00	A&SBX-2	3.90E-07	2.75E-02	+	2.58E-02
DH&BLX-0.5	3.47E-11	3.01E+00	+	1.71E+00	DH&BLX-0.5	4.80E-11	1.78E-02		1.74E-02
DH&BLX-0.3	1.02E-05	5.98E-05	+	3.80E-05	DH&BLX-0.3	7.69E-05	1.39E-02		1.43E-02
DH&SBX-5	1.01E-06	3.49E-06	+	1.67E-06	DH&SBX-5	8.76E-06	1.16E-02		1.29E-02
DH&SBX-2	5.30E-05	2.26E-05	+	1.03E-05	DH&SBX-2	3.45E-05	1.40E-02		1.68E-02
DH&FR-0.5	1.99E-11	1.69E+00	+	1.36E+00	DH&FR-0.5	9.21E-11	1.16E-02		1.99E-02
DH&2P	8.49E-11	1.32E-01		4.24E-01	DH&2P	6.28E-10	1.97E-02	+	1.55E-02
DH&U	7.32E-11	4.65E-10	+	3.45E-10	DH&U	2.53E-10	1.19E-02		1.47E-02

Table 16 Results for the expansion of F10 and the system of linear equations

EF10	B	A	T	SD	SLE	B	A	T	SD
A	1.66E+00	3.37E+00	+	1.79E+00	A	2.84E+00	2.50E+01		1.89E+01
G	1.36E+01	5.83E+01	+	2.64E+01	G	5.55E+00	5.31E+01	+	5.27E+01
BLX-0.3	1.58E-01	3.18E-01	+	1.21E-01	BLX-0.3	1.44E+00	2.03E+01		2.16E+01
BLX-0.5	6.74E+00	1.47E+01	+	4.54E+00	BLX-0.5	1.42E+00	2.62E+01		2.69E+01
SBX-2	3.21E+00	1.35E+01	+	8.26E+00	SBX-2	5.40E-01	3.54E+01		3.82E+01
SBX-5	2.54E+00	1.99E+01	+	1.46E+01	SBX-5	8.03E+00	1.14E+02	+	8.52E+01
FR-0.5	1.54E-01	2.45E-01	+	7.29E-02	FR-0.5	3.53E+00	2.66E+01		1.72E+01
2P	5.45E-01	1.60E+00	+	8.98E-01	2P	8.36E+01	2.82E+02	+	1.55E+02
U	1.14E+00	2.70E+00	+	1.16E+00	U	6.76E+01	3.68E+02	+	2.00E+02
DH	1.74E-01	1.31E+00	+	8.92E-01	DH	5.62E+01	1.27E+02	+	5.19E+01
A&U	7.13E-01	1.34E+00	+	4.44E-01	A&U	2.08E+00	3.40E+01	+	2.32E+01
G&U	6.85E-01	1.35E+01	+	2.29E+01	G&U	2.98E+00	6.67E+01	+	5.35E+01
U&BLX-0.5	1.34E-01	2.37E-01	+	8.69E-02	U&BLX-0.5	4.97E+00	5.17E+01	+	5.40E+01
U&BLX-0.3	9.56E-02	2.22E-01	+	9.02E-02	U&BLX-0.3	2.77E+00	6.34E+01	+	4.88E+01
U&SBX-5	1.23E-01	5.95E-01	+	4.93E-01	U&SBX-5	4.70E+01	2.20E+02	+	1.29E+02
U&SBX-2	1.54E-01	6.25E-01	+	4.53E-01	U&SBX-2	4.25E+00	1.03E+02	+	7.67E+01
U&FR-0.5	5.31E-02	1.74E-01	+	9.47E-02	U&FR-0.5	1.79E+00	8.07E+01	+	8.41E+01
A&2P	9.13E-01	1.45E+00	+	4.87E-01	A&2P	8.27E-01	3.89E+01	+	2.72E+01
G&2P	7.09E-01	1.88E+01	+	1.69E+01	G&2P	5.42E+00	7.70E+01	+	5.84E+01
2P&BLX-0.5	1.07E-01	2.63E-01	+	1.21E-01	2P&BLX-0.5	4.59E+00	3.30E+01		3.41E+01
2P&BLX-0.3	1.41E-01	3.01E-01	+	1.21E-01	2P&BLX-0.3	5.38E+00	4.81E+01	+	4.82E+01

Table 16 (Contd.)

EF10	B	A	T	SD	SLE	B	A	T	SD
2P&SBX-5	2.92E-01	2.74E+00	+	3.95E+00	2P&SBX-5	4.41E+01	2.07E+02	+	1.39E+02
2P&SBX-2	5.26E-01	1.26E+00	+	7.33E-01	2P&SBX-2	1.42E+01	1.00E+02	+	9.56E+01
2P&FR-0.5	7.93E-02	2.17E-01	+	9.50E-02	2P&FR-0.5	3.27E+00	6.92E+01	+	6.02E+01
A&BLX-0.5	3.79E-01	8.29E-01	+	2.72E-01	A&BLX-0.5	2.04E+00	1.80E+01	**	1.38E+01
A&BLX-0.3	4.70E-01	1.30E+00	+	4.31E-01	A&BLX-0.3	1.75E+00	2.47E+01		1.88E+01
A&SBX-5	1.12E+00	4.97E+00	+	3.86E+00	A&SBX-5	1.43E+00	2.15E+01		1.84E+01
A&SBX-2	1.06E+00	2.59E+00	+	1.07E+00	A&SBX-2	2.65E+00	2.83E+01		2.15E+01
DH&BLX-0.5	5.98E-02	1.08E-01	**	4.37E-02	DH&BLX-0.5	5.69E+00	4.25E+01	+	3.45E+01
DH&BLX-0.3	1.52E+00	2.06E+00	+	3.29E-01	DH&BLX-0.3	1.24E+00	3.36E+01		2.81E+01
DH&SBX-5	1.15E+00	2.04E+00	+	9.49E-01	DH&SBX-5	7.06E+00	1.03E+02	+	7.74E+01
DH&SBX-2	1.49E+00	2.55E+00	+	7.02E-01	DH&SBX-2	3.73E+00	7.22E+01	+	6.47E+01
DH&FR-0.5	3.88E-02	1.11E-01		6.48E-02	DH&FR-0.5	6.58E+00	5.91E+01	+	4.17E+01
DH&2P	1.22E-01	8.53E-01	+	7.87E-01	DH&2P	4.76E+01	1.28E+02	+	7.17E+01
DH&U	8.28E-02	6.41E-01	+	8.05E-01	DH&U	1.76E+01	1.15E+02	+	6.31E+01

Table 17 Results for Rosenbrock's function and the polynomial fitting problem

Rosenbrock	B	A	T	SD	PFP	B	A	T	SD
A	2.13E+01	2.25E+01	+	3.98E-01	A	2.01E+01	1.97E+02		1.24E+02
G	2.21E+01	2.27E+01	+	1.80E-01	G	2.65E+01	3.45E+02	+	2.84E+02
BLX-0.3	1.92E+01	2.18E+01		7.35E-01	BLX-0.3	3.53E+01	2.19E+02		1.55E+02
BLX-0.5	2.09E+01	2.61E+01		1.42E+01	BLX-0.5	1.95E+01	3.16E+02		2.58E+02
SBX-2	1.74E+01	2.99E+01		1.98E+01	SBX-2	3.99E+01	4.18E+02	+	2.85E+02
SBX-5	1.64E+00	3.90E+01	+	2.71E+01	SBX-5	4.58E+01	8.03E+02	+	8.99E+02
FR-0.5	1.56E+01	2.54E+01		1.53E+01	FR-0.5	6.06E+00	4.51E+02	+	3.38E+02
2P	1.31E-01	4.70E+01	+	3.18E+01	2P	6.21E+02	4.77E+03	+	3.22E+03
U	1.60E+00	5.10E+01	+	2.96E+01	U	3.60E+02	4.56E+03	+	5.50E+03
DH	1.99E+01	2.17E+01		5.70E-01	DH	1.14E+02	7.40E+02	+	4.65E+02
A&U	2.12E+01	2.23E+01	+	3.79E-01	A&U	2.42E+01	2.35E+02		1.82E+02
G&U	2.23E+01	2.26E+01	+	1.21E-01	G&U	8.71E+01	5.05E+02	+	4.01E+02
U&BLX-0.5	1.92E+01	2.97E+01		1.90E+01	U&BLX-0.5	1.00E+01	4.80E+02	+	4.23E+02
U&BLX-0.3	1.72E+01	2.30E+01		9.72E+00	U&BLX-0.3	7.70E+00	4.27E+02	+	3.28E+02
U&SBX-5	4.20E-02	2.97E+01		2.47E+01	U&SBX-5	4.85E+01	1.43E+03	+	8.44E+02
U&SBX-2	1.00E+01	3.38E+01		2.36E+01	U&SBX-2	4.83E+01	8.46E+02	+	1.01E+03
U&FR-0.5	6.07E+00	4.06E+01	+	2.60E+01	U&FR-0.5	7.22E+01	9.87E+02	+	6.95E+02
A&2P	2.13E+01	2.24E+01	+	5.09E-01	A&2P	4.80E+01	3.20E+02	+	2.01E+02
G&2P	2.22E+01	2.26E+01	+	1.33E-01	G&2P	7.66E+01	5.97E+02	+	4.07E+02
2P&BLX-0.5	1.65E+01	3.23E+01		2.29E+01	2P&BLX-0.5	2.82E+01	5.22E+02	+	4.26E+02
2P&BLX-0.3	1.70E+01	2.45E+01		1.16E+01	2P&BLX-0.3	2.39E+01	5.29E+02	+	4.62E+02
2P&SBX-5	2.50E+00	3.89E+01	+	2.59E+01	2P&SBX-5	5.81E+01	1.87E+03	+	1.79E+03
2P&SBX-2	1.47E+01	2.89E+01		1.99E+01	2P&SBX-2	3.26E+01	1.19E+03		1.93E+03
2P&FR-0.5	4.02E+00	3.44E+01		2.49E+01	2P&FR-0.5	1.31E+01	8.08E+02	+	6.28E+02
A&BLX-0.5	2.17E+01	2.23E+01	+	2.23E-01	A&BLX-0.5	2.70E+01	1.91E+02		1.66E+02
A&BLX-0.3	1.96E+01	2.24E+01	+	5.79E-01	A&BLX-0.3	2.34E+01	1.74E+02		1.22E+02
A&SBX-5	2.08E+01	2.22E+01	+	4.47E-01	A&SBX-5	8.10E+01	2.74E+02	+	1.52E+02
A&SBX-2	2.06E+01	2.21E+01	+	4.68E-01	A&SBX-2	6.09E+01	2.52E+02		1.75E+02
DH&BLX-0.5	1.68E+01	2.14E+01		9.33E-01	DH&BLX-0.5	5.95E+00	2.65E+02		2.33E+02
DH&BLX-0.3	1.91E+01	2.14E+01		8.65E-01	DH&BLX-0.3	7.51E+00	2.16E+02		1.39E+02
DH&SBX-5	1.11E+01	2.58E+01		1.46E+01	DH&SBX-5	2.45E+01	4.20E+02	+	4.31E+02
DH&SBX-2	1.65E+01	2.12E+01	**	1.26E+00	DH&SBX-2	3.22E+01	3.79E+02		3.93E+02
DH&FR-0.5	1.70E+01	2.64E+01		1.56E+01	DH&FR-0.5	1.80E+01	4.08E+02	+	3.10E+02
DH&2P	1.45E+01	2.64E+01		1.59E+01	DH&2P	3.41E+02	1.21E+03	+	1.39E+03
DH&U	1.57E+01	2.69E+01		1.68E+01	DH&U	1.69E+02	7.41E+02	+	4.00E+02

Table 18 Results for the FMS parameter identification and Ackley's function

FMSPI	B	A	T	SD	Ackley	B	A	T	SD
A	1.17E+01	2.14E+01	+	3.28E+00	A	2.43E-04	5.13E-04	+	1.93E-04
G	3.66E-07	1.85E+01	+	6.46E+00	G	2.77E-04	5.55E-04	+	1.69E-04
BLX-0.3	9.65E-13	1.39E+01	+	6.75E+00	BLX-0.3	1.45E-05	3.92E-05	+	1.52E-05
BLX-0.5	3.40E-12	1.50E+01	+	4.56E+00	BLX-0.5	2.42E-03	1.02E-02	+	6.38E-03
SBX-2	1.15E+01	1.79E+01	+	4.05E+00	SBX-2	1.17E-04	2.27E-04	+	8.51E-05
SBX-5	4.36E-15	1.08E+01	+	4.98E+00	SBX-5	2.95E-05	9.26E-05	+	4.54E-05
FR-0.5	2.84E-15	7.30E+00	**	6.67E+00	FR-0.5	7.18E-06	1.81E-05	+	6.44E-06
2P	8.71E-09	1.01E+01		7.79E+00	2P	1.14E-04	2.54E-04	+	8.31E-05
U	1.15E-07	1.14E+01		6.63E+00	U	1.88E-04	4.19E-04	+	1.52E-04
DH	1.03E-11	1.64E+01	+	7.91E+00	DH	1.52E-07	3.81E-07	**	1.67E-07
A&U	1.17E-09	1.53E+01	+	8.52E+00	A&U	1.37E-04	3.10E-04	+	1.11E-04
G&U	4.16E-13	1.35E+01	+	7.89E+00	G&U	1.68E-04	3.27E-04	+	1.15E-04
U&BLX-0.5	2.33E-14	1.06E+01		5.77E+00	U&BLX-0.5	1.08E-05	1.91E-05	+	5.56E-06
U&BLX-0.3	5.55E-15	8.79E+00		7.64E+00	U&BLX-0.3	1.47E-05	3.49E-05	+	1.29E-05
U&SBX-5	5.83E-15	9.02E+00		6.20E+00	U&SBX-5	1.45E-05	2.86E-05	+	1.21E-05
U&SBX-2	1.01E-12	1.26E+01	+	5.59E+00	U&SBX-2	1.15E-05	4.11E-05	+	1.93E-05
U&FR-0.5	9.13E-16	7.48E+00		6.27E+00	U&FR-0.5	3.14E-06	1.09E-05	+	5.54E-06
A&2P	5.70E-10	1.70E+01	+	6.04E+00	A&2P	1.46E-04	2.95E-04	+	9.73E-05
G&2P	1.05E-11	1.53E+01	+	6.68E+00	G&2P	1.45E-04	3.06E-04	+	1.17E-04
2P&BLX-0.5	4.52E-15	9.71E+00		6.30E+00	2P&BLX-0.5	6.22E-06	1.80E-05	+	5.32E-06
2P&BLX-0.3	1.86E-16	9.95E+00		6.33E+00	2P&BLX-0.3	1.24E-05	3.32E-05	+	1.14E-05
2P&SBX-5	1.89E-14	1.10E+01		5.12E+00	2P&SBX-5	2.09E-05	5.51E-05	+	2.90E-05
2P&SBX-2	2.67E+00	1.04E+01		5.26E+00	2P&SBX-2	2.49E-05	7.62E-05	+	3.63E-05
2P&FR-0.5	9.38E-16	7.36E+00		7.35E+00	2P&FR-0.5	1.99E-06	1.13E-05	+	6.08E-06
A&BLX-0.5	9.48E-11	1.78E+01	+	5.36E+00	A&BLX-0.5	5.35E-05	1.39E-04	+	4.82E-05
A&BLX-0.3	1.13E+01	1.88E+01	+	4.42E+00	A&BLX-0.3	1.61E-04	2.82E-04	+	9.70E-05
A&SBX-5	2.63E-12	1.57E+01	+	7.03E+00	A&SBX-5	1.03E-04	1.69E-04	+	5.44E-05
A&SBX-2	1.01E+01	1.79E+01	+	4.31E+00	A&SBX-2	9.55E-05	1.72E-04	+	5.33E-05
DH&BLX-0.5	8.07E-16	1.27E+01		7.67E+00	DH&BLX-0.5	1.01E-06	2.33E-06	+	6.54E-07
DH&BLX-0.3	6.12E-08	1.17E+01		7.59E+00	DH&BLX-0.3	1.25E-03	2.37E-03	+	5.77E-04
DH&SBX-5	2.86E-09	1.10E+01		7.59E+00	DH&SBX-5	2.74E-04	5.61E-04	+	1.61E-04
DH&SBX-2	5.99E-02	9.10E+00		7.61E+00	DH&SBX-2	7.20E-04	1.35E-03	+	3.10E-04
DH&FR-0.5	2.83E-17	1.08E+01		6.96E+00	DH&FR-0.5	9.84E-07	3.07E-06	+	1.29E-06
DH&2P	4.31E-10	1.36E+01	+	7.55E+00	DH&2P	2.42E-06	6.09E-06	+	2.72E-06
DH&U	9.16E-09	1.04E+01		7.33E+00	DH&U	2.17E-06	5.17E-06	+	2.15E-06

Table 19 Results for Watson's and Bohachevsky's functions

Watson	B	A	T	SD	Bohachevsky	B	A	T	SD
A	1.11E+00	1.12E+00	+	8.41E-03	A	1.96E-12	2.29E-11	+	1.67E-11
G	1.11E+00	1.12E+00	+	1.55E-02	G	1.17E-12	2.19E-11	+	3.26E-11
BLX-0.3	1.11E+00	1.11E+00	+	2.79E-03	BLX-0.3	2.22E-16	7.52E-14	+	1.23E-13
BLX-0.5	1.11E+00	1.16E+00	+	3.50E-02	BLX-0.5	7.09E-13	7.84E-12	+	7.70E-12
SBX-2	1.11E+00	1.37E+00	+	2.74E-01	SBX-2	4.48E-14	1.74E-12	+	2.08E-12
SBX-5	1.11E+00	1.13E+00		4.73E-02	SBX-5	1.99E-15	1.91E-13		4.30E-13
FR-0.5	1.11E+00	1.11E+00	+	1.09E-02	FR-0.5	1.07E-14	7.33E-14	+	6.92E-14
2P	1.11E+00	1.11E+00	+	1.22E-02	2P	7.15E-13	4.44E-12	+	4.71E-12
U	1.11E+00	1.11E+00		3.11E-03	U	4.44E-13	2.22E-11	+	2.44E-11
DH	1.11E+00	1.11E+00	+	3.69E-03	DH	0.00E+00	0.00E+00	**	0.00E+00
A&U	1.11E+00	1.11E+00	+	7.11E-03	A&U	4.00E-13	4.31E-12	+	4.36E-12
G&U	1.11E+00	1.11E+00	+	6.36E-03	G&U	1.34E-13	5.96E-12	+	6.98E-12
U&BLX-0.5	1.11E+00	1.11E+00	+	5.24E-03	U&BLX-0.5	9.10E-15	4.91E-14	+	3.69E-14
U&BLX-0.3	1.11E+00	1.11E+00		3.29E-03	U&BLX-0.3	0.00E+00	2.09E-14	+	3.61E-14
U&SBX-5	1.10E+00	1.11E+00	+	2.85E-03	U&SBX-5	3.33E-16	3.96E-14	+	6.78E-14
U&SBX-2	1.11E+00	1.35E+00		5.05E-01	U&SBX-2	1.55E-15	6.71E-14	+	9.89E-14
U&FR-0.5	1.11E+00	1.11E+00		4.40E-03	U&FR-0.5	2.99E-15	1.56E-14	+	1.51E-14
A&2P	1.11E+00	1.11E+00	+	5.67E-03	A&2P	2.58E-13	4.16E-12	+	4.96E-12
G&2P	1.11E+00	1.11E+00	+	4.20E-03	G&2P	7.15E-13	5.07E-12	+	5.65E-12
2P&BLX-0.5	1.11E+00	1.11E+00	+	8.38E-03	2P&BLX-0.5	9.32E-15	5.66E-14	+	4.37E-14
2P&BLX-0.3	1.11E+00	1.11E+00		2.21E-03	2P&BLX-0.3	0.00E+00	2.66E-14		5.84E-14
2P&SBX-5	1.11E+00	1.11E+00		1.57E-03	2P&SBX-5	1.77E-15	4.69E-14	+	5.70E-14
2P&SBX-2	1.11E+00	1.11E+00		6.92E-03	2P&SBX-2	3.66E-15	7.07E-14	+	5.47E-14
2P&FR-0.5	1.10E+00	1.11E+00		3.77E-03	2P&FR-0.5	2.77E-15	1.78E-14	+	2.06E-14

Table 19 (Contd.)

Watson	B	A	T	SD	Bohachevsky	B	A	T	SD
A&BLX-0.5	1.11E+00	1.11E+00	+	6.24E-03	A&BLX-0.5	3.14E-14	9.42E-13	+	9.52E-13
A&BLX-0.3	1.11E+00	1.12E+00	+	9.06E-03	A&BLX-0.3	1.79E-13	5.57E-12	+	7.06E-12
A&SBX-5	1.11E+00	1.11E+00	+	6.18E-03	A&SBX-5	3.44E-14	1.32E-12	+	1.32E-12
A&SBX-2	1.11E+00	1.11E+00	+	7.26E-03	A&SBX-2	9.17E-14	1.81E-12	+	2.37E-12
DH&BLX-0.5	1.11E+00	1.11E+00		2.92E-03	DH&BLX-0.5	0.00E+00	2.22E-17		1.19E-16
DH&BLX-0.3	1.11E+00	1.11E+00	+	6.47E-03	DH&BLX-0.3	4.57E-08	2.90E-07	+	3.66E-07
DH&SBX-5	1.11E+00	1.12E+00	+	1.23E-02	DH&SBX-5	7.57E-10	6.66E-09	+	5.00E-09
DH&SBX-2	1.11E+00	1.12E+00	+	7.83E-03	DH&SBX-2	4.84E-09	5.52E-08	+	4.51E-08
DH&FR-0.5	1.11E+00	1.11E+00		2.71E-03	DH&FR-0.5	0.00E+00	6.66E-17		3.00E-16
DH&2P	1.11E+00	1.11E+00	**	1.15E-03	DH&2P	0.00E+00	2.96E-17		1.40E-16
DH&U	1.11E+00	1.11E+00		1.87E-03	DH&U	0.00E+00	7.40E-18		3.98E-17

Table 20 Results for Colville's function

Coville	B	A	T	SD
A	-1.63E+02	-1.50E+02	+	1.39E+01
G	-1.61E+02	-1.44E+02	+	1.39E+01
BLX-0.3	-1.64E+02	-1.62E+02	+	4.29E+00
BLX-0.5	-1.64E+02	-1.62E+02	+	4.71E+00
SBX-2	-9.00E+02	-8.81E+02		1.94E+01
SBX-5	-9.00E+02	-8.86E+02		1.82E+01
FR-0.5	-1.64E+02	-1.63E+02	+	3.15E+00
2P	-1.64E+02	-1.63E+02	+	2.00E+00
U	-1.64E+02	-1.61E+02	+	5.13E+00
DH	-1.64E+02	-1.62E+02	+	2.46E+00
A&U	-1.64E+02	-1.58E+02	+	6.20E+00
G&U	-1.64E+02	-1.52E+02	+	1.32E+01
U&BLX-0.5	-1.64E+02	-1.63E+02	+	3.15E+00
U&BLX-0.3	-1.64E+02	-1.64E+02	+	2.62E-12
U&SBX-5	-8.93E+02	-6.43E+02	+	1.39E+02
U&SBX-2	-9.00E+02	-8.87E+02	**	1.83E+01
U&FR-0.5	-1.64E+02	-1.64E+02	+	8.56E-13
A&2P	-1.64E+02	-1.60E+02	+	4.50E+00
G&2P	-1.64E+02	-1.53E+02	+	1.33E+01
2P&BLX-0.5	-1.64E+02	-1.64E+02	+	1.85E-11
2P&BLX-0.3	-1.64E+02	-1.64E+02	+	9.68E-12
2P&SBX-5	-8.83E+02	-6.35E+02	+	1.34E+02
2P&SBX-2	-9.00E+02	-8.84E+02		1.90E+01
2P&FR-0.5	-1.64E+02	-1.64E+02	+	4.13E-13
A&BLX-0.5	-1.64E+02	-1.61E+02	+	5.03E+00
A&BLX-0.3	-1.64E+02	-1.62E+02	+	4.29E+00
A&SBX-5	-8.59E+02	-5.04E+02	+	1.53E+02
A&SBX-2	-9.00E+02	-8.81E+02		4.94E+01
DH&BLX-0.5	-1.64E+02	-1.64E+02	+	5.68E-14
DH&BLX-0.3	-1.64E+02	-1.64E+02	+	7.03E-09
DH&SBX-5	-9.00E+02	-8.41E+02		1.51E+02
DH&SBX-2	-9.00E+02	-8.82E+02		2.09E+01
DH&FR-0.5	-1.64E+02	-1.64E+02	+	5.68E-14
DH&2P	-1.64E+02	-1.63E+02	+	1.54E+00
DH&U	-1.64E+00	-1.63E+02	+	1.48E+00