Reciprocity and Consistency of Fuzzy Preference Relations

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Summary. Preference relations are the most common representation structures of information used in decision making problems because they are useful tool in modelling decision processes, above all when we want to aggregate experts' preferences into group preferences. Therefore, to establish rationality properties to be verified by preference relations is very important in the design of good decision making models. There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations: the first one requires indifference between any alternative and itself, the second one assumes the property of reciprocity in the pairwise comparison between any two alternatives, and the third one is associated with the transitivity in the pairwise comparison among any three alternatives. Furthermore, it would also be desirable to maintain the rationality assumptions on the individual preferences in the aggregation process, so that the collective preferences verify the same ones. However, as this is not always the case, establishing conditions that guarantee the preservation of these rationality properties throughout the aggregation process becomes very important.

In this article we address this problem and present a review of the main results that we have obtained about reciprocity and consistency properties of fuzzy preference relations. In particular, we present a characterization of fuzzy consistency based on the additive transitivity property which facilitates the verification of consistency in the case of fuzzy preference relations. Using this new characterization we give a method to construct consistent fuzzy preference relations from $n-1$ given preference values. We also discuss some questions concerning the compatibility between the three levels of rationality, as well as the conflict that appears between the additive consistency property and the scale used to provide fuzzy preferences. Finally, we provide aggregation operators that provide reciprocal and consistent collective preference relations when the individual preference relations are reciprocal and consistent.

1 Introduction

It is widely acknowledged that fuzzy sets play an important role in decision making because human judgements including preferences are often vague. We should consider, for instance, the situation when a set of feasible options have to be pairwise compared. In these cases, the opinions of the experts are usually described by using preference relations. Many important decision models have been developed using mainly two kinds of preference relations: *fuzzy preference relations* [7,8,13,18,29,38,40] and *multiplicative preference relations* [8,19,22,34–36].

The classical multi-criteria decision making procedure follows two steps [18]: *aggregation* and *exploitation*. The aggregation phase defines an outranking relation which indicates the global preference between every ordered pair of alternatives, taking into consideration the weights of the different points of view. The exploitation phase transforms the global information about the alternatives into a global ranking of them. This can be done in different ways, the most common one being the use of a ranking method to obtain a score function.

In a preference relation an expert associates a real number to each pair of alternatives that reflects the preference degree, or the ratio of preference intensity, of the first alternative over, or to that of, the second one. When doing this, a first and natural question immediately arises: Which conditions have to be verified in order to obtain consistent results?.

There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations [21]:

- The first level of rationality requires indifference between any alternative and itself.
- The second one assumes the property of reciprocity in the pairwise comparison between any two alternatives.
- Finally, the third one is associated with the transitivity in the pairwise comparison among any three alternatives.

The mathematical modelling of all these rationality assumptions obviously depends on the scales used for providing the preference values [14,18,23,32,34–36,40,41,50].

A preference relation verifying the third level of rationality is called a *consistent preference relation* and any property that guarantees the transitivity of the preferences is called a consistency property. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, in fact crucial, to study conditions under which consistency is satisfied [18,23,34]. On the other hand, perfect consistency is difficult to obtain in practice, specially when measuring preferences on a set with a large number of alternatives.

Clearly, the problem of consistency itself includes two problems [4,5,27]:

(i) when an expert, considered individually, is said to be consistent and,
(ii) when a whole group of experts are considered consistent.

We will focus on both problems when the expert's preferences are expressed by means of a fuzzy preference relation defined over a finite and fixed set of alternatives.

In a crisp model, where an expert provides his/her opinion on the set of alternatives, $X = \{x_1, x_2, \ldots, x_n; n \geq 2\}$, by means of a binary preference relation, R, the concept of consistency has traditionally been defined in terms of acyclicity [37], that is the absence of sequences such as $x_1, x_2, \ldots, x_k (x_{k+1} = x_1)$ with $x_j R\, x_{j+1} \forall j = 1, \ldots, k$.

In a fuzzy context, where the expert's opinions are represented using fuzzy preference relations, a traditional requirement to characterize consistency is using transitivity, in the sense that if an alternative x_i is preferred to alternative x_j and this one to x_k then alternative x_i should be preferred to x_k. Stronger conditions have been given to define consistency, for example *max-min transitivity property* or *additive transitivity property* [18,39,40,50]. However, the problem is the difficulty to check and to guarantee such consistency conditions in the decision making processes.

Furthermore, it would be also desirable to maintain the rationality assumptions of the individual preferences in the aggregation process, so that the collective preferences verify the same ones. Therefore, to establish conditions that guarantee that rationality properties are maintained through the aggregation processes is quite important.

In this article, we review some issues on reciprocity and consistency of fuzzy preference relations in decision making, our main results being:

1. A characterization of fuzzy consistency based on the additive transitivity property which facilitates the verification of consistency in the case of fuzzy preference relations.
2. A method to construct consistent fuzzy preference relations from $n-1$ given preference values based on this new characterization.
3. Aggregation operators that guarantee the reciprocity and consistency of the collective fuzzy preference relation when the individual fuzzy preference relations are both reciprocal and consistent.
4. A compatibility study between the three levels of rationality, where we show that some of the transitivity properties suggested to define the consistency of a fuzzy preference relation are not compatible with the reciprocity property.

In order to present these results, in Section 2 we present the use of preference relations in decision making. Section 3 studies the different characterizations of consistency of fuzzy preference relations. Section 4 defines a new characterization of consistency and the constructing method of consistent fuzzy preference relations. In Section 5, we discuss some questions concerning the compatibility between the three levels of rationality. In Section 6, some results concerning the preservation of the reciprocity and consistency properties in the aggregation process of fuzzy preference relations are provided. Finally, in Section 7 we draw our conclusions.

2 The Use of Preference Relations

Preference relation are the most common representation of information used in decision making problems because they are a useful tool in modelling decision processes, above all when we want to aggregate experts' preferences into group preferences [18,34,35,39]. Many important decision models have been developed using mainly two kinds of preference relations:

1. **Multiplicative preference relations [34,35]:** A multiplicative preference relation A on a set of alternatives X is represented by a matrix $A \subset X\mathrm{x}X$, $A = (a_{ij})$, being a_{ij} interpreted as the ratio of the preference intensity of alternative x_i to that of x_j, i.e., it is interpreted as x_i is a_{ij} times as good as x_j. Saaty suggests measuring a_{ij} using a ratio scale, and precisely the 1 to 9 scale [34,35]: $a_{ij} = 1$ indicates indifference between x_i and x_j, $a_{ij} = 9$ indicates that x_i is absolutely preferred to x_j, and $a_{ij} \in \{1, \ldots, 9\}$ indicates intermediate preference evaluations. In this case, the preference relation, A, is usually assumed multiplicative reciprocal, i.e.,
$$a_{ij} \cdot a_{ji} = 1 \quad \forall i, j \in \{1, \ldots, n\}.$$
By consistency Saaty means what he calls *cardinal transitivity* in the strength of preferences which is a stronger condition than the traditional requirement of the transitivity of preferences. Thereby, the definition of consistency proposed by Saaty is the following:

   Definition 2.1. [34,35] A reciprocal multiplicative preference relation $A = (a_{ij})$ is consistent if
$$a_{ij} \cdot a_{jk} = a_{ik} \quad \forall i, j, k = 1, \ldots, n.$$

2. **Fuzzy preference relations [7,18,40]:** A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is characterized by a membership function
$$\mu_P : X \times X \longrightarrow [0, 1].$$
When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $P = (p_{ij})$ being $p_{ij} = \mu_P(x_i, x_j)$, $i, j \in \{1, \ldots, n\}$. p_{ij} is interpreted as the preference degree of the alternative x_i over x_j: $p_{ij} = 1/2$ indicates indifference between x_i and x_j ($x_i \sim x_j$), $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j, and $p_{ij} > 1/2$ indicates that x_i is preferred to x_j ($x_i \succ x_j$). In this case, the preference matrix, P, is usually assumed additive reciprocal, i.e.,
$$p_{ij} + p_{ji} = 1 \quad \forall i, j \in \{1, \ldots, n\}.$$

In [8] we studied the transformation function between reciprocal multiplicative preference relations with values in the interval scale $[1/9, 9]$ and reciprocal fuzzy preference relations with values in $[0, 1]$. This study can be summarized in the following proposition:

Proposotion 2.1. [8] Suppose that we have a set of alternatives, $X = \{x_1, \ldots, x_n\}$, and associated to it a reciprocal multiplicative preference relation $A = (a_{ij})$ with $a_{ij} \in [1/9, 9]$. Then, the corresponding reciprocal fuzzy preference relation, $P = (p_{ij})$ with $p_{ij} \in [0, 1]$, associated with A is given as follows:

$$p_{ij} = g(a_{ij}) = \frac{1}{2} \cdot (1 + \log_9 a_{ij}) \, .$$

With such a transformation function g we can relate the research issues obtained for both kinds of preference relations.

In the following section, we briefly study the different proposals used to characterize consistency of fuzzy preference relations that exist in the literature.

3 Consistency of Fuzzy Preference Relations

To make a consistent choice when assuming fuzzy preference relations a set of properties or restrictions to be satisfied by such fuzzy preference relations have been suggested. Transitivity is one of the most important properties concerning preferences, and it represents the idea that the preference value obtained by directly comparing two alternatives should be equal to or greater than the preference value between those two alternatives obtained using an indirect chain of alternatives [14,32,40]. This is expressed in the following definition:

Definition 3.1. [18,50] A fuzzy preference relation P is T-transitive, with T a t-norm, if

$$p_{ik} \geq T\left(p_{ij}, p_{jk}\right) \;\; \forall i, j, k \in \{1, 2, \ldots, n\}.$$

Some of the suggested properties are given:

1. *Triangle condition [32]*: $p_{ij} + p_{jk} \geq p_{ik} \;\; \forall i, j, k.$

   This condition can be geometrically interpreted considering alternatives $x_i,\ x_j,\ x_k$ as the vertices of a triangle with length sides $p_{ij},\ p_{jk}$ and p_{ik} [32], and therefore the length corresponding to the vertices $x_i,\ x_k$ should not exceed the sum of the lengths corresponding to the vertices $x_i,\ x_j$ and $x_j,\ x_k$.

2. *Weak transitivity [40]*: $\forall i, j, k : \min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq 0.5$.

   The interpretation of this condition is the following: If x_i is preferred to x_j and x_j is preferred to x_k, then x_i should be preferred to x_k. This kind of transitivity is the usual transitivity condition that a logical and consistent person should use if he/she does not want to express inconsistent opinion, and therefore it is the minimum requirement condition that a consistent fuzzy preference relation should meet.

3. *Max-min transitivity [14,50]*: $p_{ik} \geq \min\{p_{ij}, p_{jk}\} \quad \forall i, j, k$.

   The idea represented here is that the preference value obtained by a direct comparison between two alternatives should be equal to or greater than the minimum partial values obtained when comparing both alternatives with an intermediate one. This kind of transitivity, as we said before, has been the traditional requirement to characterize consistency in the case of fuzzy preference relations [50], although it is a very strong concept that could not be verified even when a fuzzy preference relation is considered perfectly consistent from a practical point of view. For example, let's consider a set of three alternatives $X = \{x_1, x_2, x_3\}$, such that $x_1 \prec x_2 \prec x_3$. Suppose that the opinions about these alternatives are given by the following fuzzy reciprocal preference relation

$$P = \begin{pmatrix} 0.5 & 0.1 & 0 \\ 0.9 & 0.5 & 0.4 \\ 1 & 0.6 & 0.5 \end{pmatrix}$$

   On the one hand, this matrix reflects the fact that $x_1 \prec x_2 \prec x_3$; it verifies weak transitivity and the triangle condition. On the other hand, it does not verifies max-min transitivity because $p_{13} < \min\{p_{12}, p_{23}\}$.

4. *Max-max transitivity [14,50]*: $p_{ik} \geq \max\{p_{ij}, p_{jk}\} \quad \forall i, j, k$.

   This concept represents the idea that the preference value obtained by a direct comparison between two alternatives should be equal to or greater than the maximum partial values obtained when comparing both alternatives using an intermediate one. Max-max transitivity is a stronger concept than max-min transitivity and therefore if a fuzzy reciprocal preference relation neither verify the latter nor the former.

5. *Restricted Max-min transitivity [40]*: $\forall i, j, k : \min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq \min\{p_{ij}, p_{jk}\}$.

   When a fuzzy preference relation verifies this condition the following concept is modelled: if an alternative x_i is preferred to x_j with a value p_{ij} and x_j is preferred to x_k with a value p_{ij}, then x_i should be preferred to x_k with at least an intensity of preference p_{ik} equal to the minimum of the above values. A consistent fuzzy preference relation has to verify this condition, which goes a step further than weak transitivity because it adds an extra requirement about the degrees of preferences involved. This transitivity condition is, therefore, stronger than weak transitivity but it is milder than max-min transitivity. It is easy to prove that the above fuzzy reciprocal preference relation P verifies restricted max-min transitivity.

6. *Restricted Max-max transitivity [40]*: $\forall i, j, k: \quad \min\{p_{ij}, p_{jk}\} \geq 0.5 \Rightarrow p_{ik} \geq \max\{p_{ij}, p_{jk}\}$.

   The following concept is modelled with this transitivity condition: if alternative x_i is preferred to x_j with a value p_{ij} and x_j is preferred to x_k with a value p_{ij}, then x_i should be preferred to x_k with at least an intensity of preference p_{ik} equal to the maximum of the above values. It is clear that this concept is a stronger condition than restricted max-min transitivity but milder than max-max transitivity. We agree with Tanino [40] and consider this transitivity condition a compulsory one to be verified by a consistent fuzzy preference relation. It is easy to prove that the fuzzy reciprocal preference relation P, given above, verifies restricted max-max transitivity.

7. *Multiplicative transitivity [40]*: $\frac{p_{ji}}{p_{ij}} \cdot \frac{p_{kj}}{p_{jk}} = \frac{p_{ki}}{p_{ik}} \quad \forall i, j, k$.

   Tanino in [40] introduced this concept of transitivity only in the case of being: $p_{ij} > 0 \quad \forall i, j$, and p_{ij}/p_{ji} interpreted as a ratio of the preference intensity for x_i to that of x_j, i.e., x_i is p_{ij}/p_{ji} times as good as x_j. Multiplicative transitivity also includes restricted max-max transitivity [39,40], and rewritten as

   $$p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji} \quad \forall i, j, k$$

   can be extended to the whole set of fuzzy reciprocal preference relations, i.e., when values of p_{ij} can be equal to 0.

8. *Additive transitivity [39,40]*: $(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \quad \forall i, j, k$ or equivalently $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall i, j, k$.

   Remark 1. This type of transitivity has the following interpretation: suppose we do want to establish a ranking between three alternatives x_i, x_j, and x_k. If we do not have any information about these alternatives it is natural to start assuming that we are in an indifference situation, that is, $x_i \sim x_j \sim x_k$, and therefore when giving preferences this situation is represented by $p_{ij} = p_{jk} = p_{ik} = 0.5$. Suppose now that we have a piece of information that says alternative $x_i \prec x_j$, that is $p_{ij} < 0.5$. It is clear then that p_{jk} or p_{ki} have to change otherwise there would be a contradiction because we would have $x_i \prec x_j \sim x_k \sim x_i$. If we suppose that $p_{jk} = 0.5$ then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must conclude then that x_k has to be preferred to x_i. Furthermore, as $x_j \sim x_k$ then $p_{ji} = p_{ki}$, and so $p_{ij} + p_{jk} + p_{ki} = p_{ij} + p_{jk} + p_{ji} = 1 + 0.5 = 1.5$. We have the same conclusion if $p_{ki} = 0.5$. In the case of $p_{jk} < 0.5$, then x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value p_{ki} has to be equal to or greater than p_{ji}, being equal only in the case of $p_{jk} = 0.5$ as we have seen. Interpreting the value $p_{ji} - 0.5$ as the intensity of preference of alternative x_j over x_i, then it seem reasonable to suppose the intensity of preference of x_k over x_i is equal to the sum of the intensities of preferences when using an intermediate alternative x_j, that is $p_{ki} - 0.5 = (p_{kj} - 0.5) + (p_{ji} - 0.5)$. The same reasoning can be applied in the case of $p_{jk} > 0.5$. The fuzzy reciprocal preference relation P, given above, verifies additive transitivity. It is easy to prove that additive transitivity is a stronger concept than restricted max-max transitivity [39,40].

The following diagram shows all logical relationships between the defined transitivity conditions. We note that there is no relationship between weak-transitivity and triangle condition [32],

$$\begin{array}{ccccccccc} & & & & (7) & & & & \\ & & & & \Downarrow & & & & \\ (1) & \Leftarrow & (8) & \Rightarrow & (6) & \Rightarrow & (5) & \Rightarrow & (2) \\ & & & & \Uparrow & & \Uparrow & & \\ & & & & (4) & \Rightarrow & (3) & & \end{array}$$

For a fuzzy preference relation to be considered to be consistent, it is natural to assume that it verifies some kind of additive property rather than a multiplicative property, which is assumed in the case of multiplicative preference relations. From the above list of conditions, max-min transitivity and max-max transitivity are transitivity concepts which are too strong in the sense that a preference relation considered consistent from a practical point of view, as in the example given by P, may not verify them; restricted max-min and restricted max-max transitivity concepts seem good alternatives to them. From a practical point of view, restricted max-max transitivity is even more adequate than restricted max-min transitivity. Furthermore, restricted max-max transitivity implies restricted max-min transitivity. If we want to include some kind of measure of strength of preference in the concept of transitivity, as in the case of consistency of multiplicative preference relation, then additive transitivity property would be used as the definition of a fuzzy preference relation to be consistent because it includes the idea of ordinal strength of preferences [9]. Furthermore, as it is shown in the next result, the consistency definition in the case of the multiplicative preference relations via the above transformation function g (given in proposition 2) is equivalent to the additive transitivity property.

Proposition 3.1.[23] Let $A = (a_{ij})$ be a consistent multiplicative preference relation, then the corresponding reciprocal fuzzy preference relation, $P = g(A)$ verifies additive transitivity property.

In this way, in this article we consider the following definition of a consistent fuzzy preference relation.

Definition 3.2.[23] A reciprocal fuzzy preference relation $P = (p_{ij})$ is consistent if

$$p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall\ i, j, k = 1, \ldots, n.$$

In what follows, we will use the term *additive consistency* to refer to consistency for fuzzy preference relations based on the additive transitivity property.

4 Additive Consistency

In this section we present a new characterization of the additive consistency condition, which states that to check additive consistency of a fuzzy preference relation P, it is only necessary to check those triplets of values $(i,\ j,\ k)$ verifying $i \leq j \leq k$. As a consequence of this equivalent condition, we design a method to construct consistent fuzzy preference relations from a set of n-1 preference values.

4.1 Characterization of Additive Consistency

Proposition 4.1. [23] For a reciprocal fuzzy preference relation $P = (p_{ij})$, the following statements are equivalent:

1. $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall\ i, j, k.$
2. $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall\ i < j < k.$

Proposition 4.1 can be rewritten as follows.

Proposition 4.2. [23] A fuzzy preference relation $P = (p_{ij})$ is consistent if and only if

$$p_{ij} + p_{jk} + p_{ik} = \frac{3}{2}, \quad \forall\ i \leq j \leq k.$$

The following result characterizes additive consistency.

Definition 4.3. [23] For a reciprocal fuzzy preference relation $P = (p_{ij})$, the following statements are equivalent:

1. $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall\ i < j < k,$
2. $p_{i(i+1)} + p_{(i+1)(i+2)} + \ldots + p_{(j-1)j} + p_{ji} = \frac{j-i+1}{2} \quad \forall\ i < j.$

4.2 A Method to Construct Consistent Fuzzy Preference Relations

The result presented in proposition 4.1 is very important because it can be used to construct a consistent fuzzy preference relation from the set of n-1 values $\{p_{12}, p_{23}, \ldots, p_{n-1n}\}$. In such a way, we can facilitate experts the expression of consistent preferences in the decision processes.

Example. Suppose that we have a set of four alternatives $\{x_1, x_2, x_3, x_4\}$ where we have certain knowledge to assure that alternative x_1 is slightly more important than alternative x_2, alternative x_2 is more important than x_3 and finally alternative x_3 is overwhelmingly more important than alternative x_4. Suppose that this situation is modelled by the preference values $\{p_{12} = 0.55, p_{23} = 0.65, p_{34} = 0.75\}$. Applying Proposition 5, we obtain:

$$p_{31} = 1.5 - p_{12} - p_{23} = 1.5 - 0.55 - 0.65 = 0.3,$$

$$p_{41} = 2 - p_{12} - p_{23} - p_{34} = 2 - 0.55 - 0.65 - 0.75 = 0.05,$$

$$p_{42} = 1.5 - p_{23} - p_{34} = 1.5 - 0.65 - 0.75 = 0.1,$$

$$p_{21} = 1 - p_{12} = 0.45, p_{13} = 1 - p_{31} = 0.7, p_{14} = 1 - p_{41} = 0.95,$$

$$p_{32} = 1 - p_{23} = 0.35, p_{24} = 1 - p_{42} = 0.9, p_{43} = 1 - p_{34} = 0.25.$$

We note that, if the primary values are different then we would have obtained a matrix P with entries not in the interval $[0,1]$, but in an interval $[-a, 1+a]$, being $a > 0$. In such a case, we would need to transform the obtained values using a transformation function which preserves reciprocity and additive consistency, that is a function $f : [-a, 1+a] \longrightarrow [0,1]$, verifying

1. $f(-a) = 0$.
2. $f(1+a) = 1$.
3. $f(x) + f(1-x) = 1, \ \forall x \in [-a, 1+a]$.
4. $f(x) + f(y) + f(z) = \frac{3}{2}, \ \forall x, y, z \in [-a, 1+a]$ such that $x + y + z = \frac{3}{2}$.

The linear solution verifying 1 and 2 takes the form $f(x) = \varphi \cdot x + \beta$, being $\varphi, \ \beta \in R$. This function is

$$f(x) = \frac{1}{1+2a} \cdot x + \frac{a}{1+2a} = \frac{x+a}{1+2a}$$

which verifies 3

$$f(x) + f(1-x) = \frac{x+a}{1+2a} + \frac{1-x+a}{1+2a} = \frac{x+a+1-x+a}{1+2a} = 1$$

and when $x + y + z = \frac{3}{2}$

$$f(x)+f(y)+f(z) = \frac{x+a}{1+2a} + \frac{y+a}{1+2a} + \frac{z+a}{1+2a} = \frac{x+y+z+3a}{1+2a} = \frac{3/2+3a}{1+2a} = \frac{3}{2}$$

verifies 4.

Summarizing: [23] The method to construct a consistent reciprocal fuzzy preference relation P' on $X = \{x_1, \ldots, x_n, n \geq 2\}$ from n-1 preference values $\{p_{12}, p_{23}, \ldots, p_{n-1n}\}$ presents the following steps:

1. Compute the set of preference values B as
$$B = \{p_{ji},\ i < j \wedge p_{ji} \notin \{p_{21}, p_{32}, \ldots, p_{nn-1}\}\}$$
$$p_{ji} = \frac{j-i+1}{2} - p_{ii+1} - p_{i+1i+2} \cdots - p_{j-1j}.$$
2. $P = \{p_{12}, p_{23}, \ldots, p_{n-1n}\} \cup B \cup \{1 - p_{12}, 1 - p_{23}, \ldots, 1 - p_{n-1n}\} \cup \neg B.$
3. $a = |min\{p_{ij}; p_{ij} \in P\}|$
4. The consistent fuzzy preference relation P' is obtained as $P' = f(P)$ such that
$$f : [-a, 1+a] \longrightarrow [1, 0]$$
$$f(x) = \frac{x+a}{1+2a}.$$

5 Some Questions on the Compatibility Between the Three Levels of Rationality of Fuzzy Preference Relations

Due to the hierarchy structure of the three rationality assumptions for a fuzzy preference relation, the verification of a particular level of rationality should be a necessary condition in order to verify the next level of rationality. This means that the third level of rationality, transitivity of preferences, should imply or be compatible with the second level of rationality, reciprocity of preferences, and the second level with the first one, indifference of any alternative with itself.

This necessary compatibility between the rationality assumptions can be used as a criterion for considering a particular condition modelling any one of the rationality level as adequate or inadequate. In the case of fuzzy preference relations, the indifference between any alternative, x_i, and itself is modelled by associating the preference value $p_{ii} = 0.5$. The reciprocity of fuzzy preferences is modelled using the property $p_{ij} + p_{ji} = 1$, $\forall i, j$. A necessary condition for a fuzzy preference relation to verify reciprocity should be that indifference between any alternative and itself holds. Because reciprocity property implies the indifference of preferences, we conclude that both properties are compatible. However, as we have already stated, more than one condition have been suggested for modelling the transitivity of preferences. Thus, a study of the compatibility between them and the reciprocity property would be of great help in deciding which one of them is the most adequate to model the transitivity of preferences.

In the following, we will show that max-max transitivity is not compatible with the reciprocity property. If a fuzzy preference relation verifies max-max transitivity and reciprocity then $p_{ik} \geq \max\{p_{ij}, p_{jk}\} \;\; \forall i, j, k$ and $p_{ij} = 1 - p_{ji}\} \;\; \forall i, j$, which implies:

$$1 - p_{ik} \leq 1 - \max\{p_{ij}, p_{jk}\} \;\; \forall i, j, k \Rightarrow p_{ki} \leq \min\{p_{kj}, p_{ji}\} \;\; \forall i, j, k$$

which contradicts max-max transitivity. The same conclusion can be obtained regarding max-min transitivity. Therefore both properties are not adequate properties to model the transitivity for fuzzy preference relations.

If we examine the relationship between restricted max-max transitivity and reciprocity, then we conclude that the fuzzy preference relation also has to verify the complementary restricted min-min transitivity, that is,

$$\forall i, j, k: \; \min\{p_{ij}, p_{jk}\} \leq 0.5 \Rightarrow p_{ik} \geq \min\{p_{ij}, p_{jk}\}.$$

However, nor restricted max-max transitivity nor restricted min-min transitivity imply reciprocity. For example, the following fuzzy preference relation

$$P = \begin{pmatrix} 0.5 \; 0.6 \; 0.8 \\ 0.4 \; 0.5 \; 0.7 \\ 0.1 \; 0.3 \; 0.5 \end{pmatrix}$$

verifies both restricted transitivity properties but it is not reciprocal. This does not imply that they are incompatible with the reciprocity property. In fact, a fuzzy preference relation can be reciprocal and still verify both restricted transitivity properties, as the one we would have obtained by changing the values p_{13} for 0.9 or the value p_{31} for 0.2.

If we examine the compatibility between the additive consistency property and reciprocity then we conclude that the first one implies the second one. Firstly, we show that additive consistency property implies indifference of preferences. Indeed, when $i = j = k$ additive consistency property reduces to $p_{ii} + p_{ii} + p_{ii} = 1.5 \; \forall i$ which implies $p_{ii} = 0.5 \; \forall i$. Secondly, we show that additive consistency property implies reciprocity property. If $k = i$ then additive consistency reduces to $p_{ij} + p_{ji} + p_{ii} = 1.5 \; \forall i, j$ and because we already have that $p_{ii} = 0.5 \; \forall i$ then $p_{ij} + p_{ji} = 1 \; \forall i, j$.

There are many reasons that point in the direction of considering additive consistency as an adequate property to model transitivity of fuzzy preferences. However, a conflict between the additive consistency property and the scale used for providing the preference values, i.e., the closed interval $[0, 1]$, can appear. To show this, we will use a simple example.

Let us suppose a set of three alternatives $\{x_1, x_2, x_3\}$ for which we have the following information: alternative x_1 is considerably more important than alternative x_2 and this one is demonstrably or very considerably more important than alternative x_3. Suppose that these statements are modelled using the following values $p_{12} = 0.75$ and $p_{23} = 9$ respectively. If we want to maintain the additive consistency property then we would obtain a negative value $p_{13} = 1.5 - p_{12} - p_{23} = -0.15$.

This conflict between the additive consistency property and the scale used for providing preference values suggests that a modification of this property where it acts incoherently has to be made. Because restricted max-max transitivity is the minimum condition required for a reciprocal fuzzy preference relation to be considered consistent, then the modification to introduce in the additive consistency property should maintain restricted max-max transitivity and, by reciprocity, the complementary restricted min-min transitivity. We will not address this problem in this contribution, leaving it for future research.

6 Reciprocity and Consistency in the Aggregation of Fuzzy Preference Relations

When aggregating experts' preferences, consisting of combining the individual preferences into a collective one in such a way that it summarizes or reflects all the properties contained in all the individual preferences, is a necessary and very important task to carry out when we want to obtain a final solution for multi-criteria decision making problems [15,18,30].

There exist three basic classes of aggregation operators: conjunctive operators, disjunctive operators and averaging operators [1]. Triangular norms and triangular conorms are the most common operators for the first two classes of aggregation operators, and are related to the logical "and" and "or" operators. The averaging operators are located between the *minimum* and *maximum* operators, which are the bounds of the t-norms and t-conorms, and have the property to be compensative, in the sense that low values can be compensated by high values so that the 'resulting trade-off lie between the most optimistic lower bound and the most pessimistic upper bound' [50].

Yager in [42] provided a family of averaging operators called the Ordered Weighted Averaging (OWA) operators, which are commutative, idempotent, continuous, monotonic, neutral, compensative and stable for positive linear transformations [43]. The OWA operators have been extensively implemented in the last few years in one way or another in the resolution process of different problems (see for example [2,6,8,11,12,16,20,24,25,31,46]) and have also proved to be very important in solving multi-criteria decision making problems because they allow the implementation of the concept of *fuzzy majority*, which is fundamental when looking for a final solution of consensus [3,17,26,28,29]. Usually, the concept of fuzzy majority is represented by means of *fuzzy linguistic quantifiers*, which are used to calculate the weighting vector of the OWA operator [49].

Because the collective preference relation summarizes or reflects all the properties contained in all the individual preferences, it would be desirable to maintain the rationality assumptions of the individual preferences in the aggregation process. Therefore, to establish conditions that guarantee that rationality properties are maintained through the aggregation process is of some importance. In the following subsections we provide a necessary and sufficient condition to maintain the reciprocity property when using OWA operators. Unfortunately, this is not a sufficient condition to maintain the additive consistency in the aggregation process. As an alternative option, we also analyse the use the Induced-OWA (IOWA) operators [47], and show that some of them maintain both reciprocity and additive consistency properties.

6.1 Reciprocity and Consistency Using OWA Operators

Let $\{P^1, \ldots, P^m\}$ be a set of reciprocal fuzzy preference relations, i.e., $p_{ij}^k + p_{ji}^k = 1, \forall i, j, k,\ P^c = (p_{ij}^c)$ the collective preference relation obtained using an OWA operator, ϕ_Q, guided by a linguistic quantifier Q. We have that

$$p_{ij}^c = \phi(p_{ij}^1, \ldots, p_{ij}^m) = \sum_{k=1}^{m} w_k \cdot p_{ij}^{\sigma(k)}$$

being $\sigma : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\}$ a permutation such that $p_{ij}^{\sigma(k)} \geq p_{ij}^{\sigma(k=1)}$, $\forall k = 1, \ldots, m-1$.

The following result holds:

Proposition 6.1.[9] If Q is a linguistic quantifier with membership function verifying

$$Q(1-x) = 1 - Q(x),\ \forall x$$

then the collective fuzzy preference relation, obtained by aggregating a set of additive reciprocal fuzzy preference relations, using an OWA operator guided by Q, is additive reciprocal.

In the case that Q is a non-decreasing relative fuzzy quantifier with membership function:

$$Q(x) = \begin{cases} 0 & 0 \leq x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & b < x \leq 1 \end{cases}$$

$a, b \in [0, 1]$, then we have the following results:

Proposition 6.2.[9] If Q is a relative non-decreasing linguistic quantifier with parameters a and b then the collective fuzzy preference relation, obtained by aggregating a set of additive reciprocal fuzzy preference relations, using an OWA operator guided by Q, is additive reciprocal if and only if $a + b = 1$.

Proposition 6.3.[9] If Q is a relative non-decreasing linguistic quantifier with parameters a and b such that $a + b < 1$ then the collective fuzzy preference relation, $P^c = (p_{ij}^c)$, $p_{ij}^c = \phi_Q(p_{ij}^1, \ldots, p_{ij}^n)$, obtained by aggregating a set of additive reciprocal fuzzy preference relations, using an OWA operator ϕ_Q guided by Q, verifies $p^c_{ij} + p^c_{ji} \geq 1, \ \forall i, j$.

Proposition 6.4.[9] If Q is a relative non-decreasing linguistic quantifier with parameters a and b such that $a + b > 1$ then the collective fuzzy preference relation, $P^c = (p^c_{ij})$, $p^c_{ij} = \phi_Q(p^1_{ij}, \ldots, p^n_{ij})$, obtained by aggregating a set of additive reciprocal fuzzy preference relations, using an OWA operator ϕ_Q guided by Q, verifies $p^c_{ij} + p^c_{ji} \leq 1, \ \forall i, j$.

Moreover, the bigger the value of $\mid a + b - 1 \mid$ the more distant the collective preference relation is from being additive reciprocal, in the sense that the bigger is $\mid p^c_{ij} + p^c_{ji} - 1 \mid$ [9].

6.2 Reciprocity and Consistency using IOWA Operators

In [47] Yager and Filev introduced a more general type of OWA operator, which they named the Induced Ordered Weighted Averaging operator.

Definition 6.1. [47] An IOWA operator of dimension n is a function $\Phi_W : (\Re \times \Re)^n \rightarrow \Re$, to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_n)$, such that $w_i \in [0, 1]$ and $\Sigma_i w_i = 1$, and it is defined to aggregate the set of second arguments of a list of n 2-tuples $\{\langle u_1, p_1 \rangle, \ldots, \langle u_n, p_n \rangle\}$ according to the following expression,

$$\Phi_W \left(\langle u_1, p_1 \rangle, \ldots, \langle u_n, p_n \rangle\right) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

being $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ a permutation such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}, \ \forall i = 1, \ldots, n-1$, i.e., $\langle u_{\sigma(i)}, p_{\sigma(i)} \rangle$ is the 2-tuple with $u_{\sigma(i)}$ the i-th highest value in the set $\{u_1, \ldots, u_n\}$.

In the above definition the reordering of the set of values to aggregate, $\{p_1, \ldots, p_n\}$, is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated to them, which is based upon their magnitude. Due to this use of the set of values $\{u_1, \ldots, u_n\}$, Yager and Filev called them the values of an order inducing variable and $\{p_1, \ldots, p_n\}$ the values of the argument variable [47,48]. The main difference between the OWA operator and the IOWA operator resides in the reordering step of the argument variable. In the case of OWA operator this reordering is based upon the magnitude of the values to be aggregated, while in the case of IOWA operator an order inducing variable has to be defined as the criterion to induce that reordering.

An immediate consequence of this definition is that if the order inducing variable is the argument variable then the IOWA operator is reduced to the OWA operator. For a detailed list of properties and uses of the IOWA operators the reader should consult [33,44,45,47,48].

Remark 2. In this contribution we will focus on the aggregation of numerical preferences, which is why we assume that the nature of the first argument of the IOWA operators is also numeric, although it could be linguistic [44,45,47,48].

Example. If we want to aggregate the set of 2-tuples

$$\{\langle 0.65, 0.87\rangle, \langle 0.13, 0.94\rangle, \langle 0.22, 0.75\rangle\},$$

using the fuzzy linguistic quantifier *"most of"*, with weighting vector

$$\left(\frac{1}{15}, \frac{10}{15}, \frac{4}{15}\right),$$

then we obtain

$$\Phi_{most}\left(\langle 0.65, 0.87\rangle, \langle 0.13, 0.94\rangle, \langle 0.22, 0.75\rangle\right) = \\ \tfrac{1}{15} \cdot 0.87 + \tfrac{10}{15} \cdot 0.75 + \tfrac{4}{15} \cdot 0.94 = \tfrac{12.13}{15}$$

In [10], we presented three special cases of IOWA operators for multicriteria decision making problems with fuzzy preference relations. These IOWA operators allow the introduction of some semantics or meaning in the aggregation, and therefore allow for better control over the aggregation stage. The first two act as the Weighted Average (WA) operator because the aggregation is based upon the reliability of the information sources, while the third one acts as the OWA operator because the ordering of the argument values is based upon a relative magnitude associated to each one of them.

Definition 6.2. [10] If a set of of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$, and each expert e_k has an importance degree, $\mu_I(e_k) \in [0, 1]$, associated to them, then an I-IOWA operator of dimension n, Φ_W^I, is an IOWA operator whose set of order inducing values is the set of importance degrees.

Definition 6.3. [10] If a set of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$, then a C-IOWA operator of dimension n, Φ_W^C, is an IOWA operator whose set of order inducing values is the set of consistency index values, $\{-CI^1, \ldots, -CI^m\}$, associated to the set of experts.

Definition 6.4. [10] If a set of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$ then a P-IOWA operator of dimension n, Φ_W^P, is an IOWA operator whose set of order inducing values is the set of relative preferences matrices, $\{\overline{P}^k = (\overline{p}_{ij}^k); k = 1, \ldots, m\}$.

The following result holds:

Proposition 6.5. [10] If a group of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about the alternatives, $X = \{x_1, \ldots, x_n\}$, by means of reciprocal and consistent fuzzy preference relations, $\{P^1, \ldots, P^m\}$, $p^k_{ij} + p^k_{ji} = 1, \forall i, j, k$, and if $\{u_1, \ldots, u_m\}$ is a set of order inducing (importance, consistency) values associated to the set of experts, then the collective preference relation, $P^c = (p^c_{ij})$ obtained by using an IOWA operator Φ_Q guided by a linguistic quantifier Q is also reciprocal and consistent.

Remark 3. The proof of reciprocity and consistency of the collective fuzzy preference relation is based upon the assumption that the order inducing values are unchanged.

7 Concluding Remarks

In decision making contexts, preference judgements are usually modelled by using preference relation. In order to make rational and consistent choices when dealing with preference relations a set of properties or restrictions to be satisfied by such preference relations have been suggested. There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations: the indifference between any alternative and itself, the property of reciprocity in the pairwise comparison between any two alternatives and the transitivity in the pairwise comparison among any three alternatives.

This hierarchy structure of these rationality assumptions for a fuzzy preference relation means that the verification of a particular level of rationality should be a necessary condition in order to verify the next level of rationality or that incompatibility amongst them is not permitted. This can be used as a criterion for considering a particular condition for modelling any one of the rationality level as adequate or inadequate. We have shown that max-max transitivity is not an adequate property to model transitivity because it is incompatible with the reciprocity property. We have given some justification for considering the additive transitivity property as an adequate property, although we also have shown that there exists a conflict between this and the scale used for providing the preferences.

We have characterized the additive consistency property and a method for constructing consistent fuzzy preference relations has been proposed. This method builds relations from a set of n-1 preference data provided by the decision makers. In such a way, consistency of the information provided by the decision makers can be assured in order to avoid obtaining inconsistent solutions.

Because the collective preference relation summarizes or reflects all the properties contained in all the individual preferences, it is desirable to maintain the rationality assumptions of the individual preferences in the aggregation process. Therefore, conditions that guarantee that rationality properties are maintained through the aggregation processes have been provided. In particular, a necessary and sufficient condition to maintain the reciprocity property when using OWA operators was given. Unfortunately, this is not a sufficient condition to maintain the additive consistency in the aggregation process. We also provided some IOWA operators that maintain both reciprocity and additive consistency properties.

Finally, as a future research, we will study the possible modifications of the additive transitivity property in order to overcome the conflict between it and the scale used for providing preferences, and we will also propose a procedure which uses the method given to build relations from a set of n-1 preference data to reconstruct consistently fuzzy preference relations with missing information.

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