Consistency Properties for Preference Relations: T-additive and T-multiplicative transitivity

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Abstract

Consistency of preference relations is associated with the study of the transitivity property. In this paper, we analyze the properties to be verified by a function, T, in order to obtain the value of preference of the alternative x_i over the alternative x_k when we already have the values of the preference of x_i over x_j, and of x_j over x_k. As a consequence, we define T-additive transitivity property as the consistency property for fuzzy preference relations and T-multiplicative transitivity property for the case of multiplicative preference relations.

Keywords: Decision making; Consistency; Transitivity; Fuzzy preference relations; Multiplicative preference relations.

1 Introduction

In decision making contexts, preference judgements are usually modelled by using preference relations: fuzzy preference relations or multiplicative preference relations. In order to make consistent choices when dealing with preference relations a set of properties or restrictions to be satisfied by such preference relations have been suggested. In the multiplicative model, a multiplicative preference relation is consistent when it verifies the so called multiplicative consistency property [7], while in the fuzzy model, restricted max-max transitivity is considered as the minimum condition to be satisfied by a fuzzy preference relation to be considered as consistent [11]. The results obtained in [1] imply that a fuzzy preference relation is consistent if and only if the corresponding multiplicative preference relation is. Therefore, a fuzzy preference relation is considered consistent when it verifies the so called additive consistency property [10, 11], which is equivalent to the multiplicative consistency one in the multiplicative model.

However, a conflict between the multiplicative and additive consistency properties and the scales used to assign preference values to judgements exists. Some arguments support that a change in the scales used in the multiplicative and fuzzy models seems unpractical. Therefore, we focus on the other possible solution to overcome the existing conflict, that is, modifying the actual definition of multiplicative and additive consistency properties.

In this paper we address this problem and study the properties to be verified for a preference relation to be considered a consistent one. As a result, we introduce the concepts of the T-additive consistency property in the case of fuzzy preference relations and the T-multiplicative consistency property for multiplicative preference relations, which consist of a relaxation of the additive and multiplicative consistency properties. We also provide a particular expression of such T consistency properties.

In order to do this, the rest of the paper is organized as follows. In section 2, we present an overview of the consistency properties defined for preference relations. In section 3, we show the existence of a conflict between the consistency properties and the scales used to provide preference relations. This conflict means that a a modification of the actual consistency properties for both multiplicative and fuzzy preference relations is needed. In section 4, we study properties to be verified by a function, T, in order to obtain the value of preference of the alternative x_i over the alternative x_k when we already have the values of the preference of x_i over x_j, and of x_j over x_k. As a result of this, in section 4 the concepts of T-additive and T-multiplicative transitivity properties are introduced. Finally, in section 6 we draw our conclusions.

2 Consistency Properties and Preference Relations

In a preference relation an expert associates to each pair of alternatives a real number that reflects the preference degree, or the ratio of preference intensity, of the first alternative over, or to that of, the second one. Two questions immediately arise when doing this:

- Which scale should be used to associate preference values to judgements?
- Which conditions have to be verified in order to obtain consistent results?

The answer to the first question depends on the selection model we are working with. The most well-known selection models are:

1. *Fuzzy model.* In this case, preferences are represented by a *fuzzy preference relation* P on a set of alternatives X, i.e. a fuzzy set on the product set $X \times X$, which is characterized by its membership function $\mu_P : X \times X \longrightarrow [0,1]$ [3, 9, 13]. This implies that the scale to use in the fuzzy model is the closed interval $[0,1]$.

2. *Multiplicative model.* In this case, preferences are represented using a *multiplicative preference relation*, $A = (a_{ij})$, on a set of alternatives X, being a_{ij} interpreted as the ratio of the preference intensity of alternative x_i to that of x_j. According to Miller's study [6], Saaty suggests measuring a_{ij} using as ratio scale, and precisely the $1-9$ scale [7], or more generally the closed interval $[1/9, 9]$.

With respect to the second question, we agree with Saaty [7] in the sense that lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important to study conditions under which consistency is satisfied.

In a crisp context, where an expert provides his/her opinion on the set of alternatives, $X = \{x_1, x_2, \ldots, x_n; n \geq 2\}$, by means of a binary preference relation, R, the concept of consistency has traditionally been defined in terms of acyclicity [8], that is the absence of sequences such as $x_1, x_2, \ldots, x_k (x_{k+1} = x_1)$ with $x_j R x_{j+1} \forall j = 1, \ldots, k$ in the binary relations, or absence of cycles.

In a fuzzy context, a traditional requirement to characterize consistency is using transitivity, in the sense that if an alternative x_i is preferred to alternative x_j and this one to x_k then alternative x_i should be preferred to x_k [12], although stronger conditions have been given to define consistency [5, 10, 11, 13].

In the multiplicative model, what Saaty means by *consistency* is what he calls *cardinal transitivity* in the strength of preferences, which is a stronger condition than the traditional requirement of the transitivity of preferences. Thereby, the definition of consistency proposed by Saaty is the following [7]

Definition 2.1. *A reciprocal multiplicative preference relation $A = (a_{ij})$ is consistent if*

$$a_{ij} \cdot a_{jk} = a_{ik} \quad \forall i, j, k = 1, \ldots, n.$$

Inconsistency for Saaty is a violation of proportionality which may nor entail violation of transitivity [7].

Some of the suggested properties in the case of fuzzy preference relations are:

1. *Triangle condition [5]:* $p_{ij} + p_{jk} \geq p_{ik} \ \forall i, j, k$.

2. *Weak transitivity [11]:* $p_{ij} \geq 0.5,\ p_{jk} \geq 0.5 \Rightarrow p_{ik} \geq 0.5 \ \forall i, j, k$.

3. *Max-min transitivity [2, 13]:* $p_{ik} \geq \min(p_{ij}, p_{jk}) \ \forall i, j, k$.

4. *Max-max transitivity [2, 13]:* $p_{ik} \geq \max(p_{ij}, p_{jk}) \ \forall i, j, k$.

5. *Restricted max-min transitivity [2, 11]:* $p_{ij} \geq 0.5,\ p_{jk} \geq 0.5 \Rightarrow p_{ik} \geq \min(p_{ij}, p_{jk}) \ \forall i, j, k$.

6. *Restricted max-max transitivity [11]:* $p_{ij} \geq 0.5,\ p_{jk} \geq 0.5 \Rightarrow p_{ik} \geq \max(p_{ij}, p_{jk}) \ \forall i, j, k$.

7. *Multiplicative transitivity [11]:* $\frac{p_{ji}}{p_{ij}} \cdot \frac{p_{kj}}{p_{jk}} = \frac{p_{ki}}{p_{ik}} \ \forall i, j, k$.

8. *Additive transitivity [4, 10, 11]:* $(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \ \forall i, j, k$, or equivalently $p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \ \forall i, j, k$.

In [1] we obtained the transformation function between multiplicative and fuzzy preference relations, which is given in the following result:

Proposition 2.1. *Suppose that we have a set of alternatives, $X = \{x_1, \ldots, x_n\}$, and associated with it a multiplicative reciprocal preference relation $A = (a_{ij})$, with $a_{ij} \in [1/9, 9]$ and $a_{ij} \cdot a_{ji} = 1, \forall i, j$. Then the corresponding fuzzy reciprocal preference relation, $P = (p_{ij})$, associated to A, with $p_{ij} \in [0, 1]$ and $p_{ij} + p_{ji} = 1, \forall i, j$, is given as follows:*

$$p_{ij} = f(a_{ij}) = \frac{1}{2}\left(1 + \log_9 a_{ij}\right).$$

The above transformation function is bijective and, therefore, allows us to transpose concepts that have been defined for fuzzy preference relations to multiplicative preference relations. Indeed, applying the above function we show that additive transitivity property for fuzzy preference relation can be seen as the parallel concept of Saaty's multiplicative consistency property:

Proposition 2.2. *Suppose that $A = (a_{ij})$ is a multiplicative consistent preference relation. Then, the corresponding reciprocal fuzzy preference relation, $P = f(A)$, associated with A, being $p_{ij} = f(a_{ij}) = \frac{1}{2} \cdot (1 + \log_9 a_{ij})$ verifies additive transitivity property.*

All this lead us to define the concept of consistent fuzzy preference relation as in the following definition [4]:

Definition 2.2. *A fuzzy reciprocal preference relation $P = (p_{ij})$ is consistent if*

$$p_{ij} + p_{jk} + p_{ki} = \frac{3}{2} \quad \forall i, j, k = 1, \ldots, n.$$

In what follows, we will use the term *additive consistency property* to refer to this consistency for fuzzy preference relations.

3 Consistency Properties and Scales Conflict

Let us suppose a set of three alternatives $\{x_1, x_2, x_3\}$ of which we have the following information: alternative x_1 is strongly more important than alternative x_2 and this one is demonstrably or very strongly more important than alternative x_3. The application of Saaty's 1-9 scale in this case gives us the values $a_{12} = 5$ and $a_{23} = 7$ respectively.

On the one hand, to maintain the multiplicative consistency property $a_{13} = a_{12} \cdot a_{23} = 35$ and thus the consistent reciprocal multiplicative preference relation should be

$$A = \begin{pmatrix} 1 & 5 & 35 \\ 1/3 & 1 & 7 \\ 1/35 & 1/5 & 1 \end{pmatrix}.$$

We observe that the application of the multiplicative consistency property results in obtaining values outside the scale $[1/9, 9]$.

On the other hand, if we restrict the possible values of a_{13} to be in $[1/9, 9]$, then from the above information it is clear that alternative x_1 has to be considered at least as very strongly more important than alternative x_3, and thus a_{13} should be greater or equal to 7. The *consistency ratio (CR)* was defined by Saaty for measuring inconsistency, with a threshold of 0.10 to accept a reciprocal multiplicative preference relation as consistent. In our example, if $a_{13} = 7$ we get a C.R. value of 0.25412, if $a_{13} = 8$ the C.R. value is 0.212892 while a C.R. value of 0.179714 is obtained with $a_{13} = 9$. In any case, the application of the scale results in the impossibility of obtaining a consistent reciprocal multiplicative preference relation for this particular situation, which should not be the case.

In order to avoid such a type of conflict, we could proceed by choosing a different scale for providing judgements or by modifying the above definition. With respect to the first question, the use of any other scale of the form $[1/a, a]$, $a \in \mathbb{R}^+$, would not make this conflict to disappear, what means that the only possible solution to overcome this conflict would consists of using the scale of pairwise comparison from 0 to $+\infty$. However, as Saaty points out in [7], this assumes that the human judgement is capable of comparing the relative dominance of any two objets, which is not the case. All these considerations mean that if we do not change the scale to be used to associate preference values to judgements then the above definition of consistency property should be modified.

Obviously, a similar analysis in the case of working with the fuzzy model can be carried out concluding that the same conflict also exists. In the next section, we will study the general conditions to be verified by a function $T : [0,1] \times [0,1] \longrightarrow [0,1]$ so that it can be used to obtain the preference value of the alternative x_i over the alternative x_k, p_{ik}, from the preference values of x_i over x_j and of x_j over x_k, $\{p_{ij}, p_{jk}\}$.

4 The value of p_{ij} with known (p_{ik}, p_{kj})

Suppose that the pair of alternatives x_i and x_k, can not be compared directly but have an alternative x_j of which we know the exact values of p_{ij} and p_{kj}. In such a case, a broad comparison between alternatives x_i and x_k on the basis of the values p_{ij} and p_{kj} can be established. Indeed, we can distinguish the following cases:

1. $p_{ij} = 0.5$ ($p_{jk} = 0.5$) what means that $x_i \sim x_j$ ($x_j \sim x_k$) and as a consequence the strength of preference between x_i and x_k should be the same as the one between x_j and x_k. We then have that : $p_{ik} = p_{jk}$ ($p_{ik} = p_{jk}$).

2. $p_{ij} > 0.5$ and $p_{jk} > 0.5$. In this case, alternative x_i is preferred to alternative x_j ($x_i \succ x_j$) and alternative x_j is preferred to alternative x_k ($x_j \succ x_k$). We then have that $x_i \succ x_j \succ x_k$ what implies $x_i \succ x_k$ and therefore $p_{ik} > 0.5$. Furthermore, in these cases restricted max-max transitivity should be imposed, what means that x_i should be preferred to x_k with a degree of intensity at least equal to the maximum of the intensities p_{ij} and p_{jk}: $p_{ik} \geq max\{p_{ij}, p_{jk}\}$, where the equality holds only when there exist indifference between at least one of the alternatives and x_j, i.e., $p_{ij} = 0.5$ or $p_{jk} = 0.5$, as we have said in case 1. As a result, in this case it should be verified: $p_{ik} > \max\{p_{ij}, p_{jk}\}$.

3. $p_{ij} > 0.5$ and $p_{jk} < 0.5$ which is equivalent to $p_{ij} > 0.5$ and $p_{kj} = 1 - p_{jk} > 0.5$, that is: $x_i \succ x_j$ and $x_k \succ x_j$. The comparison of alternatives x_i and x_k has to done by comparing the intensities of preferences of them both over the alternative x_j. An indifference situation between x_i and x_k would exist when the preference degree of both alternatives are preferred over x_j with the same intensity, while the alternative with greatest intensity of preference over alternative x_j should be preferred to the other one. This is summarized in the following:

$$p_{ik} = 0.5 \Leftrightarrow p_{ij} = p_{kj} \Leftrightarrow p_{ij} + p_{jk} = 1$$
$$p_{ik} > 0.5 \Leftrightarrow p_{ij} > p_{kj} \Leftrightarrow p_{ij} + p_{jk} > 1$$
$$p_{ik} < 0.5 \Leftrightarrow p_{ij} < p_{kj} \Leftrightarrow p_{ij} + p_{jk} < 1$$

Moreover, it is obvious that the greater the value $|p_{ij} + p_{jk} - 1|$ the greater $|p_{ik} - 0.5|$.

4. $p_{ij} < 0.5$ and $p_{jk} > 0.5$ which is equivalent to $p_{ji} > 0.5$ and $p_{jk} > 0.5$. In this case, if alternative x_j is preferred to both alternatives x_i and x_k with the same intensity then there would be an indifference situation between them, otherwise the alternative with the lowest value of intensity of preference of alternative x_j over it would be preferred. As a consequence:

$$\begin{array}{l} p_{ik} = 0.5 \Leftrightarrow p_{ji} = p_{jk} \Leftrightarrow p_{ij} + p_{jk} = 1 \\ p_{ik} > 0.5 \Leftrightarrow p_{ji} < p_{jk} \Leftrightarrow p_{ij} + p_{jk} > 1 \\ p_{ik} < 0.5 \Leftrightarrow p_{ji} > p_{jk} \Leftrightarrow p_{ij} + p_{jk} < 1 \end{array}$$

   Again, the greater the value $|p_{ij}+p_{jk}-1|$ the greater $|p_{ik} - 0.5|$.

5. $p_{ij} < 0.5$ and $p_{jk} < 0.5$. In this case alternative x_j is preferred to alternative x_i ($x_j \succ x_i$) and alternative x_k is preferred to alternative x_j ($x_k \succ x_j$). So we have that $x_k \succ x_j \succ x_i$ what implies $x_k \succ x_i$ and therefore $p_{ik} < 0.5$. In this case, as we already said, it should be verified that $p_{ki} > \max\{p_{kj}, p_{ji}\}$ and therefore $p_{ik} < \min\{p_{ij}, p_{jk}\}$.

Cases 1 to 5 suggest that the value p_{ik} is related to the value $p_{ij} + p_{jk}$, and therefore we can assume that there exists function

$$T : [0, 1]^2 \to [0, 1],$$

such that

$$p_{ik} = T(p_{ij}, p_{jk}).$$

The above considerations mean that function T must verify:

1. $T(0.5, y) = y \ \forall y$

2. T is increasing in the interval $[0.5, 1] \times [0.5, 1]$ with respect to the value $\max\{x, y\}$ and $T(x, y) \geq \max\{x, y\}$ being equal only in the case $\min\{x, y\} = 0.5$.

3. T is increasing in the interval $[0, 0.5] \times [0, 0.5]$ with respect to the value $\min\{x, y\}$ and $T(x, y) \leq \min\{x, y\}$ being equal only in the case $\max\{x, y\} = 0.5$.

4. T is increasing in the sets $[0, 0.5) \times (0.5, 1]$ and $(0.5, 1] \times [0, 0.5)$ with respect to the value $x + y - 1$ and takes the value 0.5 when $x + y - 1 = 0$.

5. T is symmetric, $T(x, y) = T(y, x)$. Because $p_{ij} + p_{jk} = 1 \ \forall i, j$, then symmetry of T is equivalent to $T(1 - x, 1 - y) = 1 - T(x, y)$.

Another desirable property to be verified by function T should be that of continuity as it is expected that a slight change of the values of (p_{ij}, p_{jk}) should produce a slight change of the value p_{ik}.

In order to know more about function T, we start assuming that $T(x, y) = f(x + y)$, with $f : [0, 2] \to [0, 1]$ a continuous and increasing such that $f(1) = 0.5$. The linear solutions verifying this last properties take the form $f_1(z) = z/2$ and $f_2(z) = z - 0.5$.

The first linear solution gives $p_{ik} = T(p_{ij}, p_{jk}) = \frac{p_{ij}+p_{jk}}{2}$, which fails to verify property restricted max-max transitivity, because $\min\{p_{ij}, p_{jk}\} \leq \dfrac{p_{ij} + p_{jk}}{2} \leq \max\{p_{ij}, p_{jk}\}$ and in the case of $p_{ij} \to 0.5 \ \wedge \ p_{jk} \to 1 \ \ (p_{ij} \to 1 \ \wedge \ p_{jk} \to 0.5)$ we get that $p_{ik} \to 0.75$ instead of $p_{ik} \to p_{jk} = 1 \ \ (p_{ik} \to p_{ij} = 1)$.

With the second linear solution we get $p_{ik} = T(p_{ij}, p_{jk}) = p_{ij} + p_{jk} - \frac{1}{2}$, which coincides with additive transitivity. In this case T verifies restricted max-max transitivity but fails to verify $p_{ij} \to 0 \wedge p_{jk} \to 0$ $(p_{ij} \to 1 \wedge p_{jk} \to 1) \ \Rightarrow \ p_{ik} \to 0$ $(p_{ik} \to 1)$

In fact, if $x, y \leq 0.5$ and $x + y \leq 0.5$ then the above function gives negative values of p_{ik}, while in the case of $x, y \geq 0.5$ and $x+y \geq 1.5$ we obtain values of p_{ik} greater than 1. We can overcome this problem by defining T as a piecewise function, with different expressions in these regions. A possible function T would be the following:

$$T(x, y) = \begin{cases} \min\{x, y\} & x + y \leq 0.5 \\ \max\{x, y\} & x + y \geq 0.5 \\ x + y - 0.5 & \text{otherwise} \end{cases}$$

A drawback of this function is that an infinite number of different cases are equally treated. For example when $p_{ij} = 0.9$ and $p_{jk} \in [0.6, 0.9]$ this function gives the value $p_{ij} = 0.9$. However, when $p_{jk} = 0.9$ alternative x_i should be preferred to x_k with a degree of intensity greater than when $p_{jk} = 0.6$. Furthermore, this function it is not continuous because $\lim_{x+y\to 1.5^-} T(x,y) = 1$ and $\lim_{x+y\to 1.5^+} T(x,y) = \max\{x,y\}$ (the same happens when $x+y \to 0.5$).

5 T-additive transitivity and T-multiplicative transitivity

A possible solution for making function T continuous would be narrowing the application of additive transitivity to the case of being $(p_{ij}, p_{jk}) \in [0, 0.5) \times (0.5, 1] \cup (0.5, 1] \times [0, 0.5)$. By doing that, we obtain a continuous function T although the same value is returned for different pairs of values $[T(0.6, 0.9) = T(0.9, 0.9) = 0.9]$. To overcome this, we propose to incorporate some kind of strength with respect to the maximum (minimum) in function T, that is:

$$T(x,y) = \begin{cases} \min\{x,y\} - h_1(x,y) & x,y \leq 0.5 \\ \max\{x,y\} + h_2(x,y) & x,y \geq 0.5 \\ x+y-0.5 & \text{otherwise} \end{cases}$$

where $h_1 : [0, 0.5]^2 \longrightarrow [0, \min\{x,y\}] \subseteq [0, 0.5]$ and $h_2 : [0.5, 1]^2 \longrightarrow [0, 1 - \max\{x,y\}] \subseteq [0, 0.5]$ are continuous and increasing functions verifying $\min\{x,y\} \to 0.5 \Rightarrow h_1(x,y) \to 0$ and $\max\{x,y\} \to 0.5 \Rightarrow h_2(x,y) \to 0$ respectively.

All these considerations allow us to define a new transitivity condition, that we name T-additive transitivity:

Definition 5.1. *A fuzzy preference relation P is T-additive transitive if*

$$p_{ik} = T(p_{ij}, p_{jk})$$

being $T : [0,1] \times [0,1] \longrightarrow [0,1]$ a function verifying:

1. $T(x,y) \geq \max\{x,y\} \quad \forall x,y \in [0.5, 1]$
2. $T(x,y) \leq \min\{x,y\} \quad \forall x,y \in [0, 0.5]$
3. $T(x,y) = x + y - 0.5$ *otherwise*

It is obvious that T-additive transitivity implies restricted max-max transitivity:

Proposition 5.1. *T-additive transitivity implies restricted max-max transitivity, restricted max-min transitivity and weak transitivity.*

In the case of working with multiplicative preference relations, T-multiplicative transitivity can be defined by using the definition of T-additive transitivity for fuzzy preference relations and the bijective transformation function that relates both preference structures. By doing that, we have:

Definition 5.2. *A multiplicative preference relation $A = (a_{ij})$ is T-multiplicative transitive if the fuzzy preference relation $P = (p_{ij})$ is T-additive transitive, with $p_{ij} = f(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij})$.*

This definition implies:

Definition 5.3. *A multiplicative preference relation A is T-multiplicative transitive if*

$$a_{ik} = \mathrm{T}(a_{ij}, a_{jk})$$

being $\mathrm{T} : [1/9, 9] \times [1/9, 9] \longrightarrow [1/9, 9]$ a function verifying:

1. $\mathrm{T}(x,y) \geq \max\{x,y\} \quad \forall x,y \in [1, 9]$
2. $\mathrm{T}(x,y) \leq \min\{x,y\} \quad \forall x,y \in [1/9, 1]$
3. $\mathrm{T}(x,y) = xy$ *otherwise*

6 Conclusions

In order to make consistent choices when dealing with preference relations a set of properties or restrictions to be satisfied by such preference relations have been suggested. In the multiplicative model, a multiplicative preference relation is consistent when it verifies the so called multiplicative consistency property. In the additive model, a fuzzy preference relation is considered consistent when it verifies the so called additive consistency property.

However, there exists a conflict between the multiplicative and additive consistency properties and the scales used to assign preference values to judgements. There exist many arguments to support that a change in the scales used in the multiplicative and fuzzy models seems unpractical and unrealistic. Therefore, the only possible solution to overcome the existing conflict seems to be a change of the definition of the consistency properties.

In this paper we have addressed this problem and have studied the properties to be verified for a preference relation to be considered a consistent one. We have introduced the concepts of T-additive consistency property in the case of fuzzy preference relations and T-multiplicative consistency property for multiplicative preference relations which consist of a relaxation of the additive and multiplicative consistency properties.

Acknowledgements

This work has been partially supported by Research Projects TIC2002-03348 and TIC2002-03276

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