An IOWA Operator Based on Additive Consistency for Group Decision Making

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Abstract

In group decision-making, experts' preferences are usually expressed by means of fuzzy preference relations. Because not all experts may be capable of maintaining consistency between all the possible pair of feasible options of the problem, it is worthwhile evaluating the degree of inconsistency of a fuzzy preference relation so that more importance can be given to the more consistent experts. To do this, a new consistency measure of a fuzzy preference relation, based on the additive consistency property, is proposed. This consistency measure is also used to propose an IOWA operator to aggregate experts' fuzzy preference relations into a collective one in such a way that high weighting values are associated to those experts with high consistency degrees.

Keywords: Additive consistency, IOWA operator, Group decision-making, majority.

1 Introduction

Decision-making is a process of selecting the best alternative from a feasible for the purpose of attaining a goal or goals. Most major decisions are made by group that may include people from different departments or from different organizations [1, 3]. Thus, many different representation formats can be used to express preferences: i) preference ordering of the alternatives, ii) utility functions, iii) fuzzy preference relations [1]. The latter has been widely used due to their effectiveness as a tool for modelling decision processes and, above all, its utility and easiness of use when we want to aggregate experts' preferences into group preferences [3, 4].

Due to the complexity of most decision-making problems, experts's preferences may not satisfy formal properties that fuzzy preference relations are assumed to verify. One of these properties, consistency, is associated with the *transitivity property.* It is obvious that consistent information, that is, information which does not imply any kind of contradiction, is more relevant than information that contains some contradictions.

A classical choice scheme for a GDM problem follows two steps before it achieves a final decision [1]: *aggregation* and *exploitation.* The aggregation step of a GDM problem consists of combining the experts' individual preferences into a group collective one in such a way that it summarizes or reflects the properties contained in all the individual preferences. The exploitation phase transforms the global information about the alternatives into a global ranking of them. This can be done in different ways, the most common one being the use of a ranking method to obtain a score function.

In this paper, we will focus on the aggregation phase of the resolution scheme for a GDM problem, where the information provided is expressed in terms of fuzzy preference relations which are not supposed to be fully consistent. In these cases, more importance should be given to the experts that provide the more *consistent* information. To do this, we will develop a new Induced Ordered Weighted Averaging (IOWA) operator to aggregate the individual preferences relations that take into account their associated consistency degrees. We call this the Additive Consistency IOWA (AC-IOWA) operator.

The rest of paper is set out as follows: section 2 briefly presents the concept of fuzzy preference relation. Section 3 deals with the issue on consistency of fuzzy preference relations, and in particular focuses on the the additive consistency, and the introduction of a measure of consistency based on it. In section 4 the aggregation of preferences using OWA and IOWA operators is presented, an the AC-IOWA operator is defined. Section 5 gives an example, and in section 6 we point out our concluding remarks and future research.

2 Fuzzy Preference Relations

In GDM problems the best alternative(s) among a finite set of feasible options, $X = \{x_1, ..., x_n\}$, $(n \geq 2)$, according to the preferences provide by a group of experts $E = \{e_1, \ldots, e_m\}$, has to be chosen. In such a decision situation, alternatives are usually pairwise compared, which makes the fuzzy preference relation the best representation format.

Definition 1. *[4] A fuzzy preference relation P on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is characterized by a membership function*

$$\mu_P : X \times X \rightarrow [0, 1]$$

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $P = (p_{ij})$ being $p_{ij} = \mu_P(x_i, x_j) \quad \forall i, j \in \{1, \ldots, n\}$ interpreted as the preference degree or intensity of the alternative x_i over x_j: $p_{ij} = 1/2$ indicates indifference between x_i and x_j $(x_i \sim x_j)$, $p_{ij} = 1$ indicates that x_i is absolutely preferred to x_j, and $p_{ij} > 1/2$ indicates that x_i is preferred to x_j $(x_i \succ x_j)$. Based on this interpretation we have that $p_{ii} = 1/2 \quad \forall i \in \{1, \ldots, n\}$ $(x_i \sim x_i)$.

As can be seen from the previous definition, an expert only needs to provide every p_{ij} value to efficiently express its criteria over the set of alternatives. However, as it will be shown in the next section, this information does not guarantee that certain "basic" properties that are usually assumed for fuzzy preference relations are satisfied.

3 On Consistency of Fuzzy Preference Relations

In terms of fuzzy preference relations, full consistency implies no contradiction between any preference values. When an expert has to express his preferences over a set of n alternatives, he has to provide n^2 preference values, and to maintain full consistency between all of them can be a difficult task.

3.1 Additive Consistency

Definition 1 does not imply any kind of consistency of a fuzzy preference relation. In fact, preferences expressed by an expert in a fuzzy preference relation can be contradictory. As studied in [2], for making a rational choice, a set of properties to be satisfied by such fuzzy preference relations have been suggested. One of these properties is *additive transitivity* [5]:

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5) \quad \forall i, j, k \in \{1, \ldots, n\} \qquad (1)$$

or equivalently:

$$p_{ij} + p_{jk} + p_{ki} = 1.5 \quad \forall i, j, k \in \{1, \ldots, n\}. \qquad (2)$$

This type of transitivity has the following interpretation: suppose we do want to establish a ranking between three alternatives x_i, x_j and x_k. If we do not have any information about these alternatives it is natural to start assuming that we are in an indifference situation, that is, $x_i \sim x_j \sim x_k$, and therefore when giving preferences this situation is represented by $p_{ij} = p_{jk} = p_{ik} = 0.5$. Suppose now that we have a piece of information that says alternative $x_i \prec x_j$, that is $p_{ij} < 0.5$. It is clear that p_{jk} or p_{ki} have to change, otherwise there would be a contradiction, because we would have $x_i \prec x_j \sim x_k \sim x_i$. If we suppose that $p_{jk} = 0.5$ then we have the situation: x_j is preferred to x_i and there is no difference in preferring x_j to x_k. We must conclude then that x_k has to be preferred to x_i. Furthermore, as $x_j \sim x_k$ then $p_{ji} = p_{ki}$, and so $p_{ij} + p_{jk} + p_{ki} = p_{ij} + p_{jk} + p_{ji} = 1 + 0.5 = 1.5$. We have the same conclusion if $p_{ki} = 0.5$. In the case of being $p_{jk} < 0.5$, then we have that x_k is preferred to x_j and this to x_i, so x_k should be preferred to x_i. On the other hand, the value p_{ki} has to be equal or greater than p_{ji}, being equal only in the case of $p_{jk} = 0.5$ as we have seen. Interpreting the value $p_{ji} - 0.5$ as the intensity of preference of alternative x_j over x_i, then it seems reasonable to suppose that the intensity of preference of x_k over x_i should be equal to the sum of the intensities of preferences when using and intermediate alternative x_j, that is, $p_{ki} - 0.5 = (p_{kj} - 0.5) + (p_{ji} - 0.5)$. The same reasoning can be applied in the case of $p_{jk} > 0.5$.

From the previous equations we obtain the following expression:

$$p_{ij} + p_{jk} - 0.5 = p_{ik} \quad \forall i, j, k \in \{1, \ldots, n\}. \qquad (3)$$

A fuzzy preference relation is additive consistent when for every three options in the problem $x_i, x_j, x_k \in X$ their associated preference degrees p_{ij}, p_{jk}, p_{ik} satisfy (3). An additive consistent fuzzy preference relation will be referred as consistent throughout the paper, as this is the only transitivity property we are considering.

3.2 Consistency Measure

Additive consistency can be used to calculate the value of a preference degree p_{ik} using other preference degrees in a fuzzy preference relation. Indeed,

$$cp_{ik}^{j} = p_{ij} + p_{jk} - 0.5 \qquad (4)$$

where cp_{ik}^j means the calculated value of p_{ik} via j, that is, using p_{ij} and p_{jk}. Obviously, if the information provided in a fuzzy preference relation is completely consistent then cp_{ik}^j $(\forall j)$ and p_{ik} coincide. However, the information given by an expert usually does not satisfy (3), because the information provided by an expert usually suffers from a certain degree of inconsistency. In these cases, the value

$$\varepsilon p_{ik} = \frac{\sum_{\substack{j=1 \\ j \neq i,k}}^{n} \left| cp_{ik}^j - p_{ik} \right|}{n-2} \tag{5}$$

can be used to measure the error expressed in a preference degree between two options. This error can be interpreted as the consistency level between the preference degree p_{ik} and the rest of the preference values of the fuzzy preference relation. Clearly, when $\varepsilon p_{ik} = 0$ then there is no inconsistency at all between p_{ij} and the other preference values, and the higher the value of εp_{ik} the more inconsistent is p_{ik} with respect to the rest of information.

The *consistency level* for the whole fuzzy preference relation P is defined as follows:

$$CL_P = \frac{\sum_{\substack{i,k=1 \\ i \neq k}}^{n} \varepsilon p_{ik}}{n^2 - n}. \tag{6}$$

When $CL_P = 0$, then the preference relation P is fully additive consistent, otherwise, the higher CL_P the more inconsistent is P.

4 A new IOWA Operator Based on Additive Consistency

The collective fuzzy preference relation, which indicates the global preference between all pairs of alternatives, is obtained by aggregating all the individual preferences. To do this, several different aggregation operators have been proposed, including the OWA [6] and IOWA [8, 9] operators.

4.1 OWA and IOWA operators

A fundamental aspect of the OWA operators is the reordering of the arguments to aggregate, based upon the magnitude of their respective values:

Definition 2. *[6] An OWA operator of dimension n is a function $\phi\colon \mathbb{R}^n \to \mathbb{R}$, that has associated to it a set of weights or weighting vector $W = (w_1, \ldots, w_n)$ such that, $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, and is defined to aggregate a list of values $\{p_1, \ldots, p_n\}$ according to the following expression,*

$$\phi(p_1, \ldots, p_n) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

being σ a permutation of $\{1, \ldots, n\}$ such that $p_{\sigma(i)}$ is the i-th highest value in the set $\{p_1, \ldots, p_n\}$.

In the process of quantifier guided aggregation, given a collection of n criteria represented as fuzzy subsets of the alternatives X, the OWA operator has been used to implement the concept of fuzzy majority in the aggregation phase by means of a *fuzzy linguistic quantifier* which indicates the proportion of criteria 'necessary for a good solution' [7]. This implementation is done by using the quantifier to calculate the OWA weights. When a fuzzy quantifier Q is used to compute the weights of the OWA operator ϕ, then it is symbolized by ϕ_Q.

In [7], Yager proposed a procedure to evaluate the overall satisfaction of Q important (u_k) criteria (experts) (e_k) by the alternative x. In this procedure, once the satisfaction values to be aggregated have been ordered, the weighting vector associated to an OWA operator using a linguistic quantifier Q are calculated following the expression

$$w_i = Q\left(\frac{\sum_{k=1}^{i} u_{\sigma(k)}}{T}\right) - Q\left(\frac{\sum_{k=1}^{i-1} u_{\sigma(k)}}{T}\right) \tag{7}$$

being $T = \sum_{k=1}^{n} u_k$ the total sum of importance, and σ the permutation used to produce the ordering of the values to be aggregated. This approach for the inclusion of importance degrees associates a zero weight to those expert with zero importance degree.

In the OWA aggregation the weights are not associated with the arguments bur with the order position of the arguments. However, as it happens with fuzzy preference relations, the values to be aggregated cannot be directly compared, or sometimes, a different order for the arguments is preferred to be used. To allow different orders in an OWA operator, Yager and Filev defined the Induced OWA operator as follows:

Definition 3. *[8, 9] An IOWA operator of dimension n is a function $\Phi_W\colon (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R}$, to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_n)$, such that $w_i \in [0,1]$ and $\Sigma_i w_i = 1$, and it is defined to aggregate the set of second arguments of a list of n 2-tuples $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ according to the following expression,*

$$\Phi_W(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)}$$

being σ a permutation of $\{1, \ldots, n\}$ such that $\langle u_{\sigma(i)}, p_{\sigma(i)}\rangle$ is the 2-tuple with $u_{\sigma(i)}$ the i-th highest value in the set $\{u_1, \ldots, u_n\}$.

In the above definition, the reordering of the set of values to be aggregated, $\{p_1, ..., p_n\}$, is induced by the reordering of the set of values $\{u_1, ..., u_n\}$ associated to them, which is based upon their magnitude. Due to this use of the set of values $\{u_1, ..., u_n\}$, Yager and Filev called them the values of an *order inducing variable* and $\{p_1, ..., p_n\}$ the values of the *argument variable* [9]. As we have mentioned, the main difference between the OWA and the IOWA operators resides on the reordering step of the argument variable. An immediate consequence of this definition is that if the order inducing variable is the argument variable then the IOWA operator is reduced to the OWA operator.

4.2 Additive Consistency IOWA operator

As said in section 3, consistent information should be given more importance in the aggregation process of a GDM problem. The general procedure for the inclusion of importance degrees in the aggregation process involves the transformation of the preference values, p^k_{ij}, under the importance degree u_k to generate a new value, $\bar{p}^k_{ij}$. This activity is carried out by means of a transformation function (t-norm operator) g, $\bar{p}^k_{ij} = g(p^k_{ij}, u_k)$.

In our case, we may as well implement the consistency degrees by an alternative approach, which consists of using them as the order inducing values of the IOWA operator. Indeed, the closer CL_P is to 0 the more consistent the information represented by P, and thus more importance should be placed on that information. In other words, we could use these values to define the ordering of the preferences to be aggregated, in which case we would be implementing the concept of consistency in the aggregation process of our decision-making. This kind of aggregation process defines an IOWA operator that we call the Additive Consistency IOWA (AC-IOWA) operator and denote it as Φ^{AC}_W.

Definition 4. *If a set of experts,* $E = \{e_1, \ldots, e_m\}$, *provides preferences about a set of alternatives,* $X = \{x_1, \ldots, x_n\}$, *by means of the fuzzy preference relations,* $\{P^1, \ldots, P^m\}$, *then the AC-IOWA operator of dimension* n, Φ^{AC}_W, *is an IOWA operator whose set of order inducing values is* $\{1 - CL_{P^1}, ..., 1 - CL_{P^m}\}$.

The application of the AC-IOWA operator, with an appropriate RIM quantifier Q function to obtain the weighting vector, will associate more important weight in the aggregation to the most consistent experts. This is illustrated in the next section.

5 Example

Suppose that three experts e_1, e_2 and e_3 provide the following fuzzy preference relations over a set of five alternatives $\{x_1, ..., x_5\}$:

$$P^1 = \begin{pmatrix} - & 0.3 & 0.0 & 0.2 & 0.2 \\ 0.7 & - & 0.2 & 0.4 & 0.3 \\ 0.9 & 0.8 & - & 0.7 & 0.8 \\ 0.8 & 0.6 & 0.3 & - & 0.6 \\ 0.8 & 0.6 & 0.2 & 0.5 & - \end{pmatrix}$$

$$P^2 = \begin{pmatrix} - & 0.3 & 0.8 & 0.4 & 0.2 \\ 0.7 & - & 0.5 & 0.6 & 0.8 \\ 0.2 & 0.6 & - & 0.4 & 0.6 \\ 0.5 & 0.3 & 0.7 & - & 0.5 \\ 0.6 & 0.6 & 0.4 & 0.5 & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.6 & 0.6 & 0.3 & 0.4 \\ 0.4 & - & 0.6 & 0.5 & 0.3 \\ 0.4 & 0.4 & - & 0.3 & 0.3 \\ 0.7 & 0.5 & 0.7 & - & 0.7 \\ 0.6 & 0.7 & 0.7 & 0.3 & - \end{pmatrix}$$

To apply the AC-IOWA operator we need to calculate the CL_P value associated with every fuzzy preference relation. To do so we must first calculate every $\epsilon p_{ij}, \forall i, j \in \{1, ..., 5\}, i \neq j$ for every preference relation. Lets take as an example the calculation of ϵp_{34} on P^1:

$$\epsilon p_{34} = \frac{|cp^1_{34} - p_{34}| + |cp^2_{34} - p_{34}| + |cp^5_{34} - p_{34}|}{3}$$

where

$$\begin{aligned} cp^1_{34} &= p_{31} + p_{14} - 0.5 = 0.9 + 0.2 - 0.5 = 0.6 \\ cp^2_{34} &= p_{32} + p_{24} - 0.5 = 0.8 + 0.4 - 0.5 = 0.7 \\ cp^5_{34} &= p_{35} + p_{54} - 0.5 = 0.8 + 0.5 - 0.5 = 0.8 \end{aligned}$$

and thus

$$\epsilon p_{34} = \frac{|0.6 - 0.7| + |0.7 - 0.7| + |0.8 - 0.7|}{3} = 0.07$$

Calculating every ϵp_{ij} for P^1 we have:

$$\epsilon P^1 = \begin{pmatrix} - & 0.0 & 0.03 & 0.0 & 0.1 \\ 0.07 & - & 0.07 & 0.03 & 0.17 \\ 0.13 & 0.07 & - & 0.07 & 0.13 \\ 0.07 & 0.03 & 0.0 & - & 0.1 \\ 0.07 & 0.03 & 0.1 & 0.03 & - \end{pmatrix}$$

Finally, the CL_P value for P^1 is obtained as the average of every ϵp_{ij}:

$$\begin{aligned} CL_{P^1} = \ & (3 \times 0.0 + 5 \times 0.03 + 3 \times 0.1 \\ & +6 \times 0.07 + 1 \times 0.17 + 2 \times 0.13) \ / \ 20 = 0.07 \end{aligned}$$

Following the same steps we can calculate each CL_P for every fuzzy preference relation:

$$CL_{P^2} = 0.28 \quad CL_{P^3} = 0.14$$

From the results obtained we can conclude that the expert e_1 is the most consistent while e_2 is the most inconsistent one.

Once every consistency level has been obtained we can aggregate experts' preferences into a global fuzzy preference relation P^{Global} using the AC-IOWA operator. In this example we will guide the IOWA operator by means of the fuzzy linguistic quantifier"most of". Yager in [7] considers the parameterized family of RIM quantifiers $Q(r) = r^\alpha,\ \alpha \geq 0$, and the particular function with $\alpha = 2$ to represent the linguistic quantifier "most of". This function is strictly increasing but when used with the IOWA or OWA operators, associates high weighting values to those experts with a low consistency value. In order to overcome this drawback, two approaches could be adopted: i) the experts are ordered using the opposite criteria, i.e. the first one being the one with lowest consistency degree, or ii) a RIM quantifier with $\alpha < 1$ is used. We choose the second one, and in particular we use the RIM function $Q(r) = r^{1/2}$ to represent the linguistic quantifier "most of". Using expression (7), the corresponding weighting vector is $(0.61, 0.24, 0.15)$:

$$\begin{aligned} P^{Global} &= \Phi^{AC}_{most}(\langle 0.07, P^1\rangle, \langle 0.28, P^2\rangle, \langle 0.14, P^3\rangle) \\ &= 0.61 \cdot P^1 + 0.24 \cdot P^3 + 0.15 \cdot P^2 \\ &= \begin{pmatrix} - & 0.37 & 0.26 & 0.25 & 0.25 \\ 0.63 & - & 0.34 & 0.45 & 0.38 \\ 0.68 & 0.67 & - & 0.56 & 0.65 \\ 0.73 & 0.53 & 0.46 & - & 0.61 \\ 0.72 & 0.65 & 0.35 & 0.45 & - \end{pmatrix} \end{aligned}$$

In the exploitation step of the resolution process, this collective fuzzy preference relation is transformed into a global ranking of the alternatives from which a final solution for the GDMP problem is obtained.

6 Concluding Remarks and Future Works

In this work we have presented an IOWA operator that gives more importance to those experts providing fuzzy preference relations with a high level of consistency. To measure the consistency level of a fuzzy preference relation, a new consistency measure based on the concept of additive transitivity has been introduced. This consistency measure has been used to define the order inducing variable for the IOWA operator. Therefore, the higher the consistency level of a fuzzy preference relation the higher it contributes to the collective fuzzy preference relation. This is a natural and rational assumption to achieve good quality solutions to GDM problems where the information provided does not comply common properties (as consistency) usually required to solve them.

In future research, a complete decision model for GDM problems with inconsistent fuzzy preference relations will be developed, in which the consistency measure will be used to estimate or reconstruct possible missing values in fuzzy preference relations.

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