Type-1 OWA operators
for aggregating fuzzy sets in decision making

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Abstract

In dealing with the problem of aggregating experts' opinions or preferences in decision making, the traditional Yager's ordered weighted averaging (OWA) operator focuses exclusively on the aggregation of crisp numbers. Unfortunately, in many cases, experts deal with vague or imprecise information or have to express their opinions on qualitative aspects that cannot be assessed by means of quantitative values. In these cases, the use of linguistic terms instead of precise numerical values seems to be more adequate. These linguistic terms can be modelled or expressed by using fuzzy sets. In this paper, we extend Yager's OWA operator and propose what we call the *type-1 OWA* operator in distinguishing from Yager's OWA operator as a way of aggregating fuzzy sets. We also introduce the concepts of *joinness* and *meetness* of type-1 OWA operators, which are fuzzy sets indicating the linguistic expressions of the degrees of compensation induced in the aggregation process. Type-1 OWA operators are therefore suitable for aggregating linguistic opinions or preferences in human decision making.

Keywords: Type-1 OWA, fuzzy sets, OWA aggregation, linguistic opinions.

1 Introduction

In human decision making, individual experts' preferences or criteria are usually combined into an overall one in such a way so that the final result of aggregation takes into account all the individual contributions [1]. This aggregation operation has become the subject of intensive research due to its practical and academic significance in integrating human opinions or multi-criteria. Currently, at least 90 different families of aggregation operators have been extensively studied [1][2][3][4], among which the Ordered Weighted Averaging (OWA) operator proposed by Yager [13] is one of the most widely used aggregators. The majority of the existing aggregation operators, including the OWA one, focus on aggregating crisp numbers. However, due to the inherent subjectivity, imprecision and vagueness in the articulation of opinions in real world decision making, human experts tend to express their opinions in a very natural way via linguistic terms, like "important", "very important", "good" and are unable to provide exact numbers [9]. As a consequence, there is a highly demand to address the problem about how to effectively aggregate decision makers' linguistic information.

Currently, there are two main schemes to aggregate linguistic information in decision making. The first scheme is to work directly on linguistic labels without considering the expressions of the linguistic terms [6][5]. The only requirement of this scheme is that these linguistic labels should satisfy an ordinal relation. One advantage of this scheme lies in its high computing efficiency due to its symbolic aggregation nature. However, one issue arising in this scheme is the precision of the linguistic operations. In some cases, this scheme may yield a solution set with multiple alternatives for decision makers. Another matter is that most existing methods of this scheme are based on the traditional Yager's OWA operator in nature which aims at aggregating crisp numbers, so the different weights used in aggregation are usually derived from the membership function of a fuzzy set that characterises a linguistic quantifier like *"most"*, which implies that uncertain linguistic labels are aggregated in terms of certain precise crisp weights.

The second scheme of aggregating linguistic informa-

tion is via the operations on the associated membership functions of the fuzzy sets [7]. However, in most methods of this scheme, the importance weights for different experts are also measured by precise numerical values, whereas in real world decision making the importance weights are usually uncertain rather than represented by crisp numbers. The fuzzy weighted averaging operator, an extensively investigated aggregation operator, was proposed to yield a linguistic evaluation on overall performance of each expert through aggregation [14], in which the aggregated objects and importance weights are all fuzzy sets.

In this paper, we generalise Yager's OWA operator and propose a new type of OWA operator named as type-1 OWA operator that can aggregate the linguistic information represented in the form of fuzzy sets. It is also shown that some existing operators of fuzzy sets can be treated as special cases of the proposed type-1 OWA operator.

It is known that there are two fuzzy set paradigms being widely used in fuzzy community, i.e., type-1 fuzzy sets[10] and type-2 fuzzy sets[9][12][8]. In this paper, the fuzzy sets addressed are type-1 fuzzy sets unless otherwise stated.

2   Definition of Type-1 OWA Operator

The departing point for suggesting type-1 OWA operators is to aggregate the linguistic variables modelled as fuzzy sets and used to express human opinions or preferences in soft decision making. Let $F(X)$ be the power set of fuzzy sets defined on the domain of discourse X. Based on Zadeh's Extension Principle, in the following we extend Yager's OWA operator by defining the type-1 OWA operator which aims at the aggregation of fuzzy sets.

Definition 1 Given the n linguistic weights $\left\{ \widetilde{W_i} \right\}_{i=1}^n$ in the form of fuzzy sets defined on the domain of discourse $U = [0, \ 1]$, a type-1 OWA operator is a mapping $\tilde{\phi}$,

$$\tilde{\phi} \ : \ F(X) \times \cdots \times F(X) \ \rightarrow \ F(X) \tag{1}$$
$$\left(\tilde{A}_1, \cdots, \tilde{A}_n \right) \ \mapsto \ \tilde{G}$$

$$\mu_{\tilde{G}}(y) = \sup_{\substack{\sum_{i=1}^n \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U \\ a_i \in X}} \frac{\mu_{\widetilde{W_1}}(w_1) * \cdots * \mu_{\widetilde{W_n}}(w_n) *}{\mu_{\tilde{A}_1}(a_1) * \cdots * \mu_{\tilde{A}_n}(a_n)} \tag{2}$$

where $*$ is a t-norm operator, $\bar{w}_i = \frac{w_i}{\sum_{i=1}^n w_i}$, and $\sigma : \{ 1, \cdots, n \} \rightarrow \{ 1, \cdots, n \}$ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall i = 1, \cdots, n-1$, i.e., $a_{\sigma(i)}$ is the ith largest element in the set $\{a_1, \cdots, a_n\}$.

From the above definition, it can be seen that the aggregating result $\tilde{\phi} \left(\tilde{A}_1, \cdots, \tilde{A}_n \right) = \tilde{G} \in F(X)$ is still a fuzzy set defined on X, it can also be expressed as

$$\tilde{\phi} \left(\tilde{A}_1, \cdots, \tilde{A}_n \right) = \int_{w_1 \in U} \cdots \int_{w_n \in U} \int_{a_1 \in X} \cdots \int_{a_n \in X} \mu_{\widetilde{W_1}}(w_1)$$
$$* \cdots * \mu_{\widetilde{W_n}}(w_n) * \mu_{\tilde{A}_1}(a_1) * \cdots * \mu_{\tilde{A}_n}(a_n) \bigg/ \sum_{i=1}^n \bar{w}_i a_{\sigma(i)} \tag{3}$$

where $\int \cdots \int$ denotes union over all admissible $\{w_i\}$ and $\{a_i\}$. In case of multiple combinations of $\{w_i\}$ and $\{a_i\}$ leading to a same aggregating point as $y = \sum_{i=1}^n \bar{w}_i a_{\sigma(i)}$, the membership grade of y is chosen to be the largest membership grade in (3). A Direct Approach to type-1 OWA operation is delineated in Table 10 given the linguistic weights $\left\{ \widetilde{W_i} \right\}_{i=1}^n$ in the form of fuzzy sets. Figure 1(right) shows four fuzzy sets and their aggregated result by the type-1 OWA operator with linguistic weights given in Figure 1(left). Following this definition, we have, for example, that $\mu_{\tilde{G}}(2.0) = 0.848$.

The way of directly expressing an extended fuzzy set at large rather than via individual points based on extension principle like in (3) has been used in fuzzy logic systems[8]. To the best of our knowledge, the way of aggregating fuzzy sets via OWA mechanism as suggested in (2) and (3) has not been reported in literature yet.

Moreover, if the linguistic weights $\left\{ \widetilde{W_i} \right\}_{i=1}^n$ and the aggregated objects $\left\{ \tilde{A}_i \right\}_{i=1}^n$ are reduced to singleton fuzzy sets, then type-1 OWA operator is reduced to the Yager's OWA operator.

3   Type-1 OWA Operator vs Some Existing Operators

Given two linguistic variables A and B with truth-values $v(A)$ and $v(B)$, in which $v(A)$ and $v(B)$ are two fuzzy sets on domain [0, 1], Zadeh in [11] defined two basic operators to carry out the aggregations of "*A and B*" and "*A or B*" by calculating the truth-values $v(A \ and \ B)$ and $v(A \ or \ B)$ respectively. Later

Table 10. A Direct Approach to Performing Type-1 OWA Operation

Step 1.

1. Set up the parameters for the membership functions of linguistic weights $\left\{ \widetilde{W}_i \right\}_{i=1}^{n}$ and aggregated objects $\left\{ \tilde{A}_i \right\}_{i=1}^{n}$.

2. Select the domains of linguistic weights, U, and that of aggregated objects, X.[1]

Step 2. $\forall\, w_1 \in U, \cdots, w_n \in U, a_1 \in X, \cdots, a_n \in X.$

1. Normalise $(w_1, \cdots, w_n)$ as

$$\bar{w}_i = w_i / \sum_{i=1}^{n} w_i$$

2. Perform Yager's OWA operation:

$$y = \phi_{\bar{w}_1, \cdots, \bar{w}_n}(a_1, \cdots, a_n)$$

3. Calculate the membership grade of the point y belonging to the $\tilde{G}$:

$$\mu_{\tilde{G}}(y) =$$
$$\mu_{\widetilde{W}_1}(w_1) * \cdots * \mu_{\widetilde{W}_n}(w_n) * \mu_{\tilde{A}_1}(a_1) * \cdots * \mu_{\tilde{A}_n}(a_n)$$

Step 3. Merge the membership grades of $\tilde{G}$ with the same point y by selecting the largest membership grade.

[1]Note: In calculation, the domain U and X are discretised, only the points on the domains with non-zero membership grades are considered, so for each fuzzy set $\widetilde{W}_i$ (or $\tilde{A}_i$), the domain U (or X) may be different.

in [12], Mizumoto and Tanaka named these two operators as the *meet* and the *join* of fuzzy sets on domain $[0, 1]$. As a matter of fact, these *meet* and *join* of fuzzy sets can be defined on a general domain X. Indeed, given two fuzzy sets $\tilde{A}$ and $\tilde{B} \in F(X)$, the meet and join operations of $\tilde{A}$ and $\tilde{B}$ are defined as follows.

Definition 2 The meet of $\tilde{A}$ and $\tilde{B}$ is

$$\tilde{A} \sqcap \tilde{B} = \int_{x \in X} \int_{y \in X} \mu_{\tilde{A}}(x) * \mu_{\tilde{B}}(y) / x \wedge y \qquad (4)$$

The join of $\tilde{A}$ and $\tilde{B}$ is

$$\tilde{A} \sqcup \tilde{B} = \int_{x \in X} \int_{y \in X} \mu_{\tilde{A}}(x) * \mu_{\tilde{B}}(y) / x \vee y \qquad (5)$$

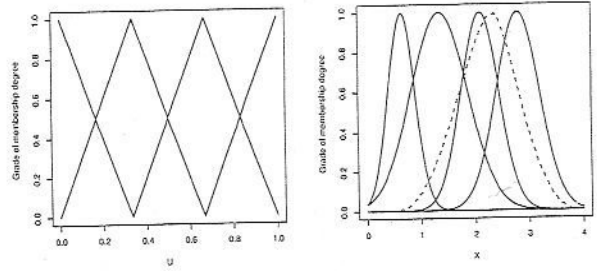where $*$ is a t-norm operator, $\wedge$ and $\vee$ represent the

Figure 1. (left)-Linguistic weights used in a type-1 OWA (from right to left): $\widetilde{W}_1$, $\widetilde{W}_2$, $\widetilde{W}_3$ and $\widetilde{W}_4$; (r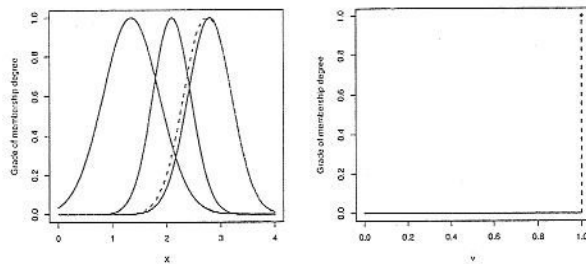ight)- Aggregation result by this type-1 OWA operator: dashed line-aggregation result; solid lines: aggregated fuzzy sets

minimum and maximum operators respectively.

Interestingly, type-1 OWA operator can be used to perform the join and/or meet operations by selecting appropriate linguistic weights in the forms of fuzzy sets. Based on extension princple, in the following, we define the degree of *joinness* and the degree of *meetness* associated with the linguistic weights of a type-1 OWA operator to characterise the degrees to which the aggregation is like the join and meet operation respectively.

Definition 3 Given a type-1 OWA operator with n linguistic weights $\left\{ \widetilde{W}_i \right\}_{i=1}^{n}$ in the form of fuzzy sets on $U = [0, \ 1]$, the *joinness* of this operator is defined as follows,

$$joinness\left(\left\{ \widetilde{W}_i \right\}_{i=1}^{n} \right) =$$
$$\int_{w_1 \in U} \cdots \int_{w_n \in U} \mu_{\widetilde{W}_1}(w_1) * \cdots * \mu_{\widetilde{W}_n}(w_n) / v \qquad (6)$$

while the *meetnes* is defined as follows,

$$meetness\left(\left\{ \widetilde{W}_i \right\}_{i=1}^{n} \right) =$$
$$\int_{w_1 \in U} \cdots \int_{w_n \in U} \mu_{\widetilde{W}_1}(w_1) * \cdots * \mu_{\widetilde{W}_n}(w_n) / (1 - v) \qquad (7)$$

where $v = \dfrac{1}{(n-1)\sum_{i=1}^{n} w_i} \sum_{i=1}^{n} (n - i) w_i$, $*$ is a t-norm.

The *joinness* and *meetness* of type-1 OWA operators are fuzzy sets indicating the linguistic expressions of

the degrees of compensations induced in their aggregations. In what follows we now provide some special classes of type-1 OWA operators. In particular, it is shown that the *meet* and *join* of fuzzy sets are particular cases of type-1 OWA operators.

3.1 Join and Join-like Operators

The join operation of fuzzy sets can be performed by applying a type-1 OWA operation with the singleton weights: $\widetilde{W}_1 = \tilde{1}$; $\widetilde{W}_i = \tilde{0}$ $(i \neq 1)$.

Indeed, equations (6) and (7) with the above weights give the values $joinness\left(\left\{\widetilde{W}_i\right\}_{i=1}^{n}\right) = \tilde{1}$, $meetness\left(\left\{\widetilde{W}_i\right\}_{i=1}^{n}\right) = \tilde{0}$, which indicate that this particular type-1 OWA operator coincides with is the join operator of fuzzy sets as illustrated in Figure 2(right). Figure 2(left) shows three fuzzy sets and their aggregation result using this particular case of type-1 OWA operator.

Figure 2. (left)-Aggregating result by the type-1 OWA operator as join: dashed line-aggregation result; solid lines: aggregated fuzzy sets; (right)-*joinness* of the type-1 OWA operator as join

Join-like type-1 OWA operators can be obtained by selecting appropriate linguistic weights. Indeed, this is the case when the first linguistic weight is close to the singleton fuzzy set $\tilde{1}$, and the rest are close to the singleton fuzzy set $\tilde{0}$ in turn, as depicted in Figure 3. The *joinness* of the type-1 OWA operator with this set of weights is illustrated in Figure 4(right), which shows that aggregation by this operator is very much like a join operation. Figure 4(left) delineates the aggregation results of three type-1 fuzzy sets by this join-like type-1 OWA operator.

3.2 Meet and Meet-like Operators

The meet operation of fuzzy sets can also be performed by type-1 OWA operation with the singleton weights: $\widetilde{W}_i = \tilde{0}$ $(i \neq n)$; $\widetilde{W}_n = \tilde{1}$. With this
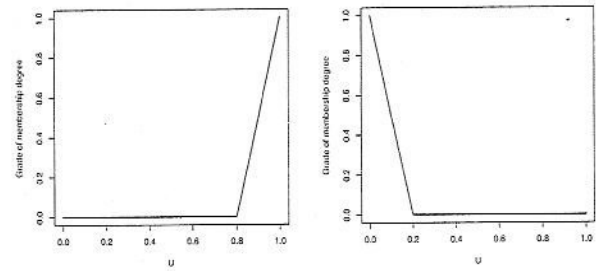

Figure 3. Join-like linguistic weights: (left) $\widetilde{W}_1$; (right) $\widetilde{W}_i$ $(i \neq 1)$
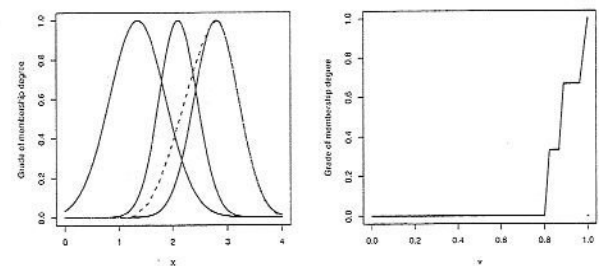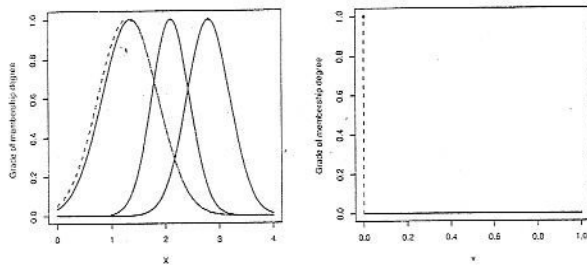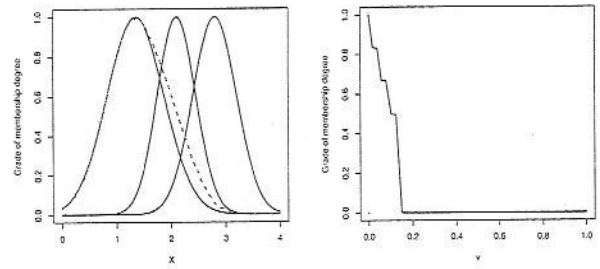

Figure 4. (left)-Aggregating result by the type-1 OWA operator with linguistic weights given in Figure 3: dashed line-aggregation result; solid lines: aggregated fuzzy sets;(right)-*joinness* of the type-1 OWA operator with linguistic weights given in Figure 3

set of weights, we have $joinness\left(\left\{\widetilde{W}_i\right\}_{i=1}^{n}\right) = \tilde{0}$, $meetness\left(\left\{\widetilde{W}_i\right\}_{i=1}^{n}\right) = \tilde{1}$ as shown in Figure 5(right). Again, these values indicate that the type-1 OWA operator defined by the above weights is the meet operator of fuzzy sets. Figure 5(left) depicts the aggregation result of three type-1 fuzzy sets by this operator.

As in the above case, meet-like operation can be obtained by type-1 OWA operator via selecting appropriate linguistic weights: the last linguistic weight approaching to $\tilde{1}$, and the rest of linguistic weights approaching to $\tilde{0}$ in turn, as depicted in Figure 6. The *joinness* of this particular type-1 OWA operator is illustrated in Figure 7(right). Figure 7(left) depicts the aggregation result of three fuzzy sets by this meet-like aggregation operator.

3.3 Mean and Mean-like operators

The mean averaging operation can be seen as the particular type of Yager's OWA operator with weights all equal to $1/n$. A type-1 OWA operator with the weights in the form of singleton fuzzy sets $\widetilde{1/n}$ can be seen as

Figure 5. (left)-Aggregating result by the type-1 OWA operator as meet: dashed line-aggregation result; solid lines: aggregated fuzzy sets; (right)-*joinness* of the type-1 OWA operator as meet

Figure 7. (left)-Aggregating result by the type-1 OWA operator with linguistic weights given in Figure 6: dashed line-aggregation result; solid lines: aggregated fuzzy sets; (right)-*joinness* of the type-1 OWA operator with linguistic weights given in Figure 6
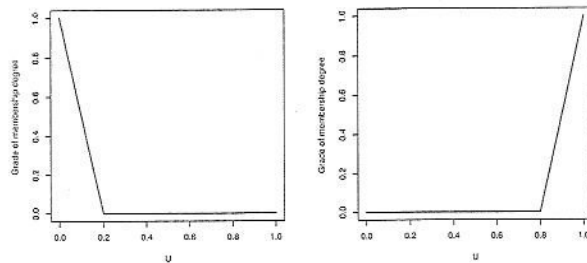
Figure 6. Meet-like linguistic weights: (left) $\widetilde{W}_i$ $(i \neq n)$; (right) $\widetilde{W}_n$
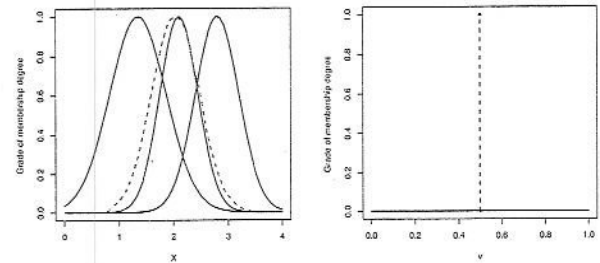
Figure 8. (left)-Aggregation with type-1 OWA operator with weights all equal to $\widetilde{1/3}$: dashed line-aggregation result; solid lines: aggregated fuzzy sets; (right)-*joinness* and *meetness* of a type-1 OWA operator with weights all equal to $\widetilde{1/3}$

the extended mean operation on fuzzy sets. For example, Figure 8(right) shows the *joinnes* and *meetness* of the type-1 OWA operator with three singleton fuzzy sets $\widetilde{1/3}$. Both *joinnes* and *meetness* are equal to $\widetilde{0.5}$ which indicate that this particular type-1 OWA operator neither has the tendency to the join operation nor the meet operation. Figure 8(left) depicts the aggregation result of three type-1 fuzzy sets by this operator.

Again, we can also define mean-like type-1 OWA operators by selecting the linguistic weights appropriately. For example, Figure 9(left) shows three linguistic weights in the forms of triangular type-1 fuzzy numbers whose cores locate at 1/3. The type-1 OWA operator with the above three linguistic weights can be used to perform a mean-like operation on fuzzy sets. The result of aggregating three fuzzy sets with this mean-like OWA operator is shown in Figure 9(right). Figure 10 illustrates the *joinness* of this type-1 OWA operator, which indicates the mean-like type-1 OWA operator neither has the tendency to the join operation nor the meet operation.

4 Conclusions and Future Work

In this paper, by generalising Yager's OWA operator, we proposed a new type of OWA aggregation operator called type-1 OWA operator in the interests of aggregating linguistic information represented by fuzzy sets. We also have introduced the concepts of *joinness* and *meetness* of type-1 OWA operators, which are fuzzy sets indicating the linguistic expressions of the degrees of compensation induced in the aggregation process. In particular we have shown that the *meet* and *join* of fuzzy sets are particular cases of type-1 OWA operators.

It is noted that the type-1 OWA operators focus on aggregation of type-1 fuzzy sets, some new topics have arisen, such as what the properties of type-1 OWA operators are, how to construct the linguistic weights, how to aggregate the type-2 fuzzy sets etc.. Moreover, this new type of OWA aggregation operator would have the great potential of being applied to
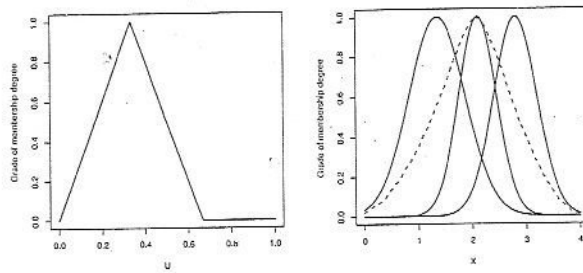
Figure 9. (left)-Linguistic weights with cores located at $1/3:\widehat{W_i}$ $(i = 1, 2, 3)$; (right)-Aggregating result by the type-1 OWA operator with linguistic weights given in (left): dashed line-aggregation result;solid lines: aggregated fuzzy sets
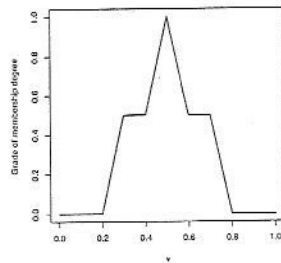

Figure 10. *joinness* of the type-1 OWA operator with linguistic weights given in Figure 9(left)

multi-expert decision making and multi-criteria decision making where the information is provided in the form of linguistic preference relations. These topics will receive much attention in our future reseach.

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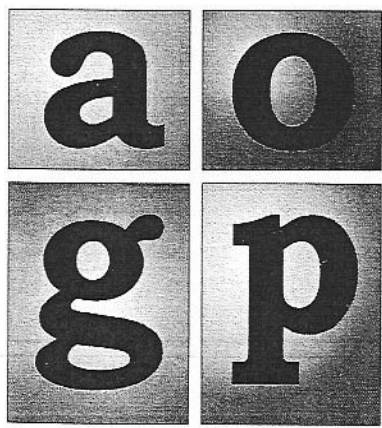
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