Cardinal Consistency of Reciprocal Preference Relations: A Characterization of Multiplicative Transitivity

Francisco Chiclana, Enrique Herrera-Viedma, Sergio Alonso, and Francisco Herrera

Abstract—**Consistency of preferences is related to rationality, which is associated with the *transitivity property*. Many properties suggested to model transitivity of preferences are inappropriate for reciprocal preference relations. In this paper, a functional equation is put forward to model the "cardinal consistency in the strength of preferences" of reciprocal preference relations. We show that under the assumptions of continuity and monotonicity properties, the set of representable uninorm operators is characterized as the solution to this functional equation. Cardinal consistency with the conjunctive representable cross ratio uninorm is equivalent to Tanino's multiplicative transitivity property. Because any two representable uninorms are order isomorphic, we conclude that multiplicative transitivity is the most appropriate property for modeling cardinal consistency of reciprocal preference relations. Results toward the characterization of this uninorm consistency property based on a restricted set of $(n-1)$ preference values, which can be used in practical cases to construct perfect consistent preference relations, are also presented.**

Index Terms—**Consistency, fuzzy preference relation, rationality, reciprocity, transitivity, uninorm.**

I. INTRODUCTION

IN ORDER to reach a decision, experts have to express their preferences by means of a set of evaluations over a set of alternatives. Different alternative preference elicitation methods were compared in [41], where it was concluded that pairwise comparison methods are more accurate than nonpairwise methods. Given two alternatives of a finite set of all potentially available X, an expert either prefers one to the other or is indifferent between them. Obviously, there is another possibility that of an expert being unable to compare them.

There exist two main mathematical models based on the concept of preference relation. In the first one, a preference relation is defined for each one of the aforementioned three possible preference states, which is usually referred to as a preference structure on the set of alternatives. The sec-

ond one integrates the three possible preference states into a single preference relation. Further to this, in each case, two different representations could be adopted: the use of binary (crisp) preference relations or the use of [0,1]-valued (fuzzy) preference relations. Reciprocal [0,1]-valued relations $(R = (r_{ij}); \forall i, j : 0 \leq r_{ij} \leq 1, r_{ij} + r_{ji} = 1)$ are frequently used in fuzzy set theory for representing intensities of preferences [2], [4], [36], [37], [44], [50]–[53]. These are the types of relations this paper deals with, and we will refer to them as simply reciprocal preference relations. In probabilistic choice theory, reciprocal preference relations describe the binary preference subsets of two alternatives of X, and are known with the name "probabilistic binary preference relations" [39].

The main advantage of pairwise comparison is that of focusing exclusively on two alternatives at a time that facilitates experts when expressing their preferences. However, this way of providing preferences limits experts in their global perception of the alternatives, generates more information than is really necessary, and, as a consequence, the provided preferences could be inconsistent.

In a crisp context, the concept of consistency has traditionally been defined in terms of acyclicity [28], [42], [43], [49]. This condition is closely related to the transitivity of the corresponding binary preference relation, in the sense that if alternative x_i is preferred to alternative x_j and this one to x_k, then alternative x_i should be preferred to x_k. In a fuzzy context, the traditional requirement to characterize consistency has followed the way of extending the classical requirements of binary preference relations. However, the main difference in this case with respect to the previous one resides on the role the intensity of preference has. Indeed, many of the properties suggested for reciprocal preference relations attempt to extend the binary notion of transitivity of preferences by implementing the intensity of preference [39]. Thus, consistency is also based on the notion of transitivity. Among these properties we can cite weak stochastic transitivity, minimum transitivity, moderate stochastic transitivity (or restricted mininum transitivity), maximum transitivity, strong stochastic transitivity (or restricted maximum transitivity), additive transitivity, and multiplicative transitivity (or product rule) [31], [33]–[35], [39], [51], [52], [56].

Recently, a general framework for studying the transitivity of reciprocal preference relations, the cycle transitivity, was presented [14]. Cycle transitivity derives as a generalization of the application of a particular *t*-norm (algebraic product) transitivity property to reciprocal preference relations. Given any three alternatives, the six possible (product transitivity) single

inequalities are transformed into a double (dual upper–lower) inequality. This result leads the authors to propose a general definition of transitivity for reciprocal preference relations: the cycle transitivity w.r.t. an upper bound function, with dual lower bound function. The authors show that stochastic (weak, moderate, and strong) transitivity properties are all special cases of cycle transitivity. Multiplicative transitivity and minimum transitivity are types of cycle transitivity w.r.t. self-dual upper bound functions. Many phenomena have also been characterized within the cycle transitivity framework, details of which can be found in [13] and [19]–[22].

We consider the term "consistency" as the "cardinal consistency in the strength of preferences" described by Saaty in [48, p. 7]:

> not merely the traditional requirement of the transitivity of preferences [...], but the actual intensity with which the preference is expressed transits through the sequence of objects in comparison.

Saaty considers a positive multiplicative reciprocal matrix ($A = (a_{ij})$; $\forall i, j : 0 < a_{ij}$, $a_{ij} \cdot a_{ji} = 1$) to be consistent if and only if $a_{ik} = a_{ij} \cdot a_{jk} \ \forall i, j, k$.

Some of the aforementioned transitivity properties are not appropriate for reciprocal preference relations. An example of this inappropriateness is the additive transitivity property that, although equivalent to Saaty's consistency property for multiplicative preference relations [6], [31], is in conflict with the $[0, 1]$ scale used for providing the preference values.

In this paper, we propose modeling the consistency of reciprocal preference relations via a functional equation: $r_{ik} = f(r_{ij}, r_{jk}) \ \forall i, j, k$. It is shown that such a function f, under reciprocity property, is associative and self-dual. Adding continuity and monotonicity properties to function f, the set of representable uninorm operators is characterized as the solution to the aforementioned functional equation. In particular, the conjunctive representable cross ratio uninorm derives Tanino's multiplicative transitivity property. Because any two representable uninorms are order-isomorphic, we conclude that multiplicative transitivity is the most appropriate property for modeling cardinal consistency of reciprocal preference relations. This result is related to the g-isostochastic transitivity concept, a special case of cycle transitivity w.r.t. self-dual upper bound functions. As De Baets *et al.* point out [14], associativity makes their function g behavior being closely related to a t-conorm on $[0.5, 1] \times [0.5, 1]$. The functional equation proposed in this paper for modeling consistency is the extension of the functional equation modeling g-isostochastic transitivity, from the restricted range $[0.5, 1] \times [0.5, 1]$ to the whole range of preference values $[0, 1] \times [0, 1]$. This range extension makes function f to become a uninorm, whose behavior on $[0, 0.5] \times [0, 0.5]$ and $[0.5, 1] \times [0.5, 1]$ is known to be closely related to t-norms and t-conorms, respectively [27], [57]. Moreover, this result means that the only g-isostochastic transitivity w.r.t. a continuous function g that can be applied to the whole range of preferences $[0, 1]$ is, up to isomorphisms, the multiplicative transitivity.

The rest of the paper is set out as follows. Section II comprises an introduction to (crisp and fuzzy) preferences and the main two approaches to model them, as well as some preliminaries on consistency of preferences. In Section III, a set of conditions and properties for a reciprocal preference relation to be considered "fully consistent" will be established. In Section IV, the set of representable uninorm operators is shown to be the solution of this set of conditions. A characterization of the functional consistency studied in Section III, based on a minimum set of $(n - 1)$ preference values, is given in Section V. This characterization can be used to construct consistent reciprocal preference relations. Finally, conclusions are drawn in Section VI.

II. PREFERENCE RELATIONS

In order for this paper to be as self-contained as possible, we include in this section a brief review of the main concepts needed throughout it. A short review of the concept of binary relations and the associated crisp numerical representations used in the literature to model preferences on a set of alternatives is presented in Section II-A, which is followed by their fuzzy extensions in Section II-B. Section II-C is devoted to a brief description of what we mean by consistency of preferences, its relation with the transitivity property of preferences, and how it has been modeled in both crisp and fuzzy cases.

A. Crisp Preference Relations

Preference relations are usually assumed to model experts' preferences in decision-making problems [24]. Given two alternatives, an expert judges them in one of the following ways: 1) one alternative is preferred to another, or 2) the two alternatives are indifferent to him/her; he/she is unable to compare them. Accordingly, three binary relations can be defined on a finite set of alternatives X.

1) The strict preference relation P: $(x_i, x_j) \in P$ if and only if the expert prefers x_i to x_j ($x_i \succ x_j$).
2) The indifference relation I: $(x_i, x_j) \in I$ if and only if the expert is indifferent between x_i and x_j ($x_i \sim x_j$).
3) The incomparability relation J: $(x_i, x_j) \in J$ if and only if the expert is unable to compare x_i and x_j.

Using a numerical representation of preferences, any ordered pair of alternatives (x_i, x_j) can be associated with a number from the set $\{0, 1\}$ as follows:

$$p_{ij} = 1 \Leftrightarrow x_i \succ x_j$$
$$p_{ij} = 0 \Leftrightarrow x_j \succ x_i.$$

In a similar way, indifference and incomparability relations can be represented numerically. A binary relation on X may be conveniently represented by a square matrix of dimension cardinality of X. We make note that a different but equivalent set of values ($\{1, -1\}$) to the earlier one has been used by Fishburn [23].

A preference structure on X is defined as a triplet (P, I, J) of binary relations on X that satisfy:

1) P is irreflexive and asymmetrical;
2) I is reflexive and symmetrical;
3) J is irreflexive and symmetrical;
4) $P \cap I = P \cap J = I \cap J = \emptyset$;
5) $P \cup P^t \cup I \cup J = X^2$.

Here, P^t is the transpose (or inverse) of P: $(x_i, x_j) \in P \Leftrightarrow$ $(x_j, x_i) \in P^t$ [47]. Condition 5) is called the completeness condition.

Fishburn [24] defines indifference as the absence of strict preference. He also points out that indifference might arise in three different ways: 1) when an expert truly feels that there is no real difference, in a preference sense, between the alternatives; 2) when the expert is uncertain as to his/her preference between the alternatives because "he might find their comparison difficult and may decline to commit himself[/herself] to a strict preference judgment while not being sure that he[/she] regards [them] equally desirable (or undesirable)"; or 3) when both alternatives are considered incomparable on a preference basis by the expert.

It is obvious from the third case that Fishburn treats the incomparability relation as an indifference relation, i.e., J is empty (there is no incomparability). Asymmetry is considered by Fishburn [24] as an "obvious" condition for preferences: if an expert prefers x_i to x_j, then he[/she] should not simultaneously prefer x_j to x_i. In this case, an alternative numerical representation to the earlier one is that of integrating the comparison outcomes preference–indifference into one single reciprocal preference relation R with the following interpretation [2]:

$$r_{ij} = 1 \Leftrightarrow x_i \succ x_j$$
$$r_{ij} = 0 \Leftrightarrow x_j \succ x_i$$
$$r_{ij} = 0.5 \Leftrightarrow x_j \sim x_i.$$

Again, a different but equivalent set of values ($\{1, 0, -1\}$) to this one has been used by Fishburn [23]. This numerical interpretation in the absence of incomparability integrates both strict preference and indifference relations in a single (reciprocal) preference relation.

Both numerical interpretations of preferences are indeed related. In [47], it is proved that a preference structure (P, I, J) on a set of alternatives X can be characterized by the single reflexive relation $\overline{R} = P \cup I$: $(x_i, x_j) \in \overline{R}$ if and only if "x_i is as good as x_j." $\overline{R}$ is called the large preference relation of (P, I, J). Conversely, given any reflexive binary relation $\overline{R}$ on X, a preference structure (P, I, J) can be constructed on it as follows: $P = \overline{R} \cap (\overline{R}^t)^c, I = \overline{R} \cap \overline{R}^t, J = R^c \cap (\overline{R}^t)^c$, where $\overline{R}^c$ is the complement of $\overline{R}$: $(x_i, x_j) \in \overline{R} \Leftrightarrow (x_j, x_i) \notin \overline{R}^c$. On the other hand, it is well known that a complete binary relation $\overline{R}$ on X, i.e., a binary relation verifying $\max\{\overline{r}_{ij}, \overline{r}_{ji}\} = 1 \, \forall i, j$, has an equivalent reciprocal preference representation R with [12], [14]

$$r_{ij} = \frac{1 + \overline{r}_{ij} - \overline{r}_{ji}}{2}.$$

B. Fuzzy and Reciprocal Preference Relations

Given three alternatives x_i, x_j, x_k such that x_i is preferred to x_j and x_j to x_k, the question whether the "degree or strength of preference" of x_i over x_j exceeds, equals, or is less than the "degree or strength of preference" of x_j over x_k cannot be answered by the classical preference modeling. The implementation of the degree of preference between alternatives may be essential in many situations. Take, for example, the case of

three alternatives $\{x, y, z\}$ and two experts. If one of the experts prefers x to y to z, and the other prefers z to y to x, then using the aforementioned numerical values, it may be difficult or impossible to decide which alternative is the best. This may not be the case if intensities of preferences are allowed in the earlier model. As Fishburn points out [23], if alternative y is closer to the best alternative than to the worst one for both experts, then it might seem appropriate to "elect" it as the social choice, while if it is closer to the worst than to the best, then it might be excluded from the choice set.

The introduction of the concept of fuzzy set as an extension of the classical concept of set when applied to a binary relation leads to the concept of a fuzzy relation. Obviously, the earlier two interpretations for modeling experts' preferences can therefore be extended to allow the implementation of intensity of preferences. In [45], we can find for the first time the fuzzy models for strict preference, indifference, and incomparability relations. The fuzzy preference structure and its characterization has been widely dealt with in the literature. For more details, the reader should consult [3], [15]–[18], [25], [26], [54], and [55]. The second fuzzy interpretation of intensity of preferences was introduced by Bezdek *et al.* [2] via the concept of a reciprocal (fuzzy) preference relation, and later reinterpreted by Nurmi [44]. The adapted definition of a reciprocal preference relation is the following.

Definition 1 (Reciprocal Preference Relation): A reciprocal preference relation R on a finite set of alternatives X is characterized by a membership function $\mu_R : X \times X \longrightarrow [0, 1], \mu(x_i, x_j) = r_{ij}$, verifying

$$r_{ij} + r_{ji} = 1 \qquad \forall i, j \in \{1, \dots, n\}.$$

When cardinality of X is small, the reciprocal preference relation may be conveniently denoted by the matrix $R = (r_{ij})$. The following interpretation is also usually assumed.

1) $r_{ij} = 1$ indicates the maximum degree of preference for x_i over x_j.
2) $r_{ij} \in \,]0.5, 1[$ indicates a definite preference for x_i over x_j.
3) $r_{ij} = 0.5$ indicates indifference between x_i and x_j.

This is the interpretation we are assuming in this paper. For more details, the reader should consult [4], [5], [36], [37], and [51]–[53].

Remark 1: As mentioned before, in probabilistic choice theory, reciprocal preference relations are referred to as probabilistic binary preference relations. In fuzzy set theory, reciprocal preference relations when used to represent intensities of preferences have usually been referred to as reciprocal fuzzy preference relations. Reciprocal preference relations can be seen as a particular case of (weakly) complete fuzzy preference relations, i.e., fuzzy preference relations satisfying $r_{ij} + r_{ji} \geq 1 \, \forall i, j$.

C. On the Consistency of Reciprocal Preference Relations

The main advantage of pairwise comparison is that of focusing exclusively on two alternatives at a time, which facilitates experts when expressing their preferences. However, this way of providing preferences limits experts in their global perception

of the alternatives, and the provided preferences could not be rational. Usually, rationality is related to consistency, which is associated with the *transitivity property* [9], [10], [31], [50]. Transitivity seems like a reasonable criterion of coherence for an individual's preferences: if x is preferred to y and y is preferred to z, common sense suggests that x should be preferred to z. Obviously, there might be some natural situations where transitivity is not appropriate to be required as pointed out by De Schuymer *et al.* [20].

There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations [32], [40].

1) The first level of rationality requires indifference between any alternative x_i and itself.
2) The second one requires that if an expert prefers x_i to x_j, that expert should not simultaneously prefer x_j to x_i. This asymmetry condition is viewed as an "obvious" condition/criterion of consistency for preferences [24]. This rationality condition is modeled by the property of reciprocity in the pairwise comparison between any two alternatives [7], which is seen by Saaty as basic in making paired comparisons [48].
3) Finally, the third one is associated with the transitivity in the pairwise comparison among any three alternatives.

A preference relation verifying the third level of rationality is usually called a *consistent preference relation* and any property that guarantees the transitivity of the preferences is called a consistency property. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, in fact crucial, to study conditions under which consistency is satisfied [48].

Consistency of reciprocal preference relations is therefore based on the notion of transitivity, in the sense that if alternative x_i is preferred to alternative x_j ($r_{ij} \geq 0.5$) and this one to x_k ($r_{jk} \geq 0.5$), then alternative x_i should be preferred to x_k ($r_{ik} \geq 0.5$). This transitivity notion is normally referred to as *weak stochastic transitivity* [39], [52]. However, the implementation of the intensity of preference in modeling consistency of reciprocal preference relations has been proposed in many different ways [31], [33]–[35], [48], [52]. Among the many properties or conditions suggested, we can cite the following.

1) *Minimum transitivity* [53]:

$$r_{ik} \geq \min\{r_{ij}, r_{jk}\} \qquad \forall i, j, k$$

2) *Moderate stochastic transitivity* (or restricted minimum transitivity) [39], [51]–[53]:

$$\forall i, j, k : \min\{r_{ij}, r_{jk}\} \geq 0.5 \Rightarrow r_{ik} \geq \min\{r_{ij}, r_{jk}\}.$$

3) *Maximum transitivity* [53]:

$$r_{ik} \geq \max\{r_{ij}, r_{jk}\} \qquad \forall i, j, k.$$

4) *Strong stochastic transitivity* (or restricted maximum transitivity) [39], [51]–[53]:

$$\forall i, j, k : \min\{r_{ij}, r_{jk}\} \geq 0.5 \Rightarrow r_{ik} \geq \max\{r_{ij}, r_{jk}\}.$$

5) *Additive transitivity* [52]:

$$(r_{ij} - 0.5) + (r_{jk} - 0.5) = r_{ik} - 0.5 \qquad \forall i, j, k.$$

6) *Multiplicative transitivity* (or product rule) [39], [51]–[53]:

$$\forall i, j, k : r_{ij}, r_{jk}, r_{ki} \notin \{0, 1\} \Rightarrow$$

$$r_{ij} \cdot r_{jk} \cdot r_{ki} = r_{ik} \cdot r_{kj} \cdot r_{ji}.$$

The first observation is that maximum transitivity cannot be verified under reciprocity. Indeed, if $R = (r_{ij})$ is reciprocal and verifies maximum transitivity, then

$$r_{ki} = 1 - r_{ik} \leq 1 - \max\{r_{ij}, r_{jk}\} = \min\{r_{ji}, r_{kj}\} \quad \forall i, j, k.$$

From maximum transitivity, we have $r_{ki} \geq \max\{r_{kj}, r_{ji}\}$ $\forall i, j, k$. Therefore, maximum transitivity and reciprocity are verified only when $r_{ik} = r_{ij} = r_{jk} = 0.5 \ \forall i, j, k$.

Additive transitivity, although equivalent to Saaty's consistency property for multiplicative preference relations [6], [31], is in conflict with the [0,1] scale used for providing the preference values, and therefore, it is an inappropriate property to model consistency of reciprocal preference relations [8].

Minimum transitivity for reciprocal preference relations has been characterized by De Baets *et al.* in their general framework of transitivity of reciprocal preference relations, the cycle transitivity [14].

Definition 2 (Cycle Transitivity): A reciprocal preference relation $R = (r_{ij})$ is called cycle-transitive w.r.t. an upper bound function U if and only if

$$L(\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}) \leq \alpha_{ijk} + \beta_{ijk} + \gamma_{ijk} - 1$$
$$\leq U(\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}) \quad \forall i, j, k$$

where $\alpha_{ijk} = \min\{r_{ij}, r_{jk}, r_{ki}\}, \beta_{ijk} = \text{median}\{r_{ij}, r_{jk}, r_{ki}\}$, $\gamma_{ijk} = \max\{r_{ij}, r_{jk}, r_{ki}\}$, and L is the dual lower function of U, i.e.

$$L(\alpha_{ijk}, \beta_{ijk}, \gamma_{ijk}) = 1 - U(1 - \gamma_{ijk}, 1 - \beta_{ijk}, 1 - \alpha_{ijk}).$$

De Baets *et al.* proved that minimum transitivity of reciprocal preference relations was the only transitivity property with 1-Lipschitz commutative conjunctor such that its corresponding upper bound is self-dual [14].

Strong stochastic transitivity under reciprocity property can be equivalently stated as

$$\begin{cases} r_{ik} \geq \max\{r_{ij}, r_{jk}\}, & \text{if } r_{ij}, r_{jk} \geq 0.5; \\ r_{ik} \leq \min\{r_{ij}, r_{jk}\}, & \text{if } r_{ij}, r_{jk} \leq 0.5; \\ \min\{r_{ij}, r_{jk}\} \leq r_{ik} \leq \max\{r_{ij}, r_{jk}\}, & \text{otherwise.} \end{cases}$$

Multiplicative transitivity property was proposed by Tanino for reciprocal preference relations when $r_{ij} > 0 \ \forall i, j$. This property has also been proposed to model transitivity of preferences in probabilistic choice theory with the name of product rule [39]. By simple algebraic manipulation, multiplicative transitivity property, under the assumption of reciprocity, can be expressed as

$$r_{ik} = \frac{r_{ij} \cdot r_{jk}}{r_{ij} \cdot r_{jk} + (1 - r_{ij}) \cdot (1 - r_{jk})}.$$

Comparison of independent random variables in terms of winning probabilities sometimes results in multiplicative transitive reciprocal relations, for instance, for random variables with an exponential distribution, power-law distribution, or Gumbel distribution [21]. De Baets *et al.* [14] have shown that multiplicative transitivity property is a special case of the cycle transitivity property w.r.t. a self-dual upper bound function.

In practical cases, we have to face the problem of which transitivity property to use for modeling and measuring the consistency of reciprocal preference relations. Obviously, apart from maximum transitivity and additive transitivity, all the earlier listed properties of transitivity could be used to address this issue. The rest of the paper is devoted to the modeling of the "cardinal consistency in the strength of preferences" for reciprocal preference relations via a functional equation. We will show that under reasonable conditions, multiplicative transitivity property is characterized, up to isomorphism, as the most appropriate property to model and measure consistency of reciprocal preference relations.

III. CARDINAL CONSISTENCY OF RECIPROCAL PREFERENCE RELATIONS

The assumption of experts being able to quantify their preferences in the domain [0,1] instead of $\{0,1\}$ underlies unlimited computational abilities and resources from the experts. Taking these unlimited computational abilities and resources into account, we may formulate that an expert's preferences are consistent when for any three alternatives x_i, x_j, x_k, their preference values are related in the following "exact" form:

$$r_{ik} = f(r_{ij}, r_{jk}) \qquad \forall i, j, k \qquad (1)$$

f being a function $f : [0,1] \times [0,1] \to [0,1]$. This functional consistency is the extension of the g-isostochastic transitivity property from $[0.5, 1]$ to the whole range of preferences $[0,1]$.

In practical cases, the aforementioned functional consistency might obviously not be verified even when a reciprocal preference relation verifies weak transitivity property. However, the assumption of modeling consistency using expression (1) can be used to introduce levels of consistency, which, in group decision-making situations, could be exploited by assigning a relative importance weight to each one of the experts in arriving to a collective preference opinion. Also, expression (1) can be used as a principle for deriving missing values. Indeed, using just these preference values provided by an expert, expression (1) could be used to estimate those preference values which were not given by that expert because he/she was uncertain as to his/her preference between the alternatives or he/she was unable to compare them. By doing this, the estimated values are assured to be "compatible" with the rest of the information provided by that expert [29]–[31].

Note that (1) implies that $f(r_{ij}, r_{jk}) = f(r_{il}, r_{lk}) \ \forall i, j, k, l$. On the other hand, we have $r_{ij} = f(r_{il}, r_{lj})$ and $r_{lk} = f(r_{lj}, r_{jk})$. Putting these expressions together we have that f is associative

$$f(f(r_{il}, r_{lj}), r_{jk}) = f(r_{il}, f(r_{lj}, r_{jk})) \qquad \forall i, j, k, l.$$

Proposition 1 (Associativity): Let $R = (r_{ij})$ be a reciprocal preference relation. A function $f : [0,1] \times [0,1] \to [0,1]$ verifying $r_{ik} = f(r_{ij}, r_{jk}) \ (\forall i, j, k)$ is associative, i.e.

$$f(f(x,y), z) = f(x, f(y,z)) \qquad \forall x, y, z \in [0,1].$$

Equation (1) and the assumed reciprocity property of preferences imply that

$$r_{ki} = f(r_{kj}, r_{ji}) = f(1 - r_{jk}, 1 - r_{ij}) \qquad \forall i, j, k.$$

Because $r_{ki} = 1 - r_{ik} = 1 - f(r_{ij}, r_{jk}) \ \forall i, j, k$, then

$$f(1 - r_{jk}, 1 - r_{ij}) = 1 - f(r_{ij}, r_{jk}) \qquad \forall i, j, k.$$

This means that function f verifies a self-duality property.

Proposition 2 (Self-Duality): Let $R = (r_{ij})$ be a reciprocal preference relation. A function $f : [0,1] \times [0,1] \to [0,1]$ verifying $r_{ik} = f(r_{ij}, r_{jk}) \ (\forall i, j, k)$ is self-dual, i.e.

$$f(x,y) + f(1 - y, 1 - x) = 1 \qquad \forall x, y \in [0,1].$$

From this result, we have

$$f(x, 1 - x) = 0.5 \qquad \forall x \in [0,1]$$

and

$$f(0.5, r_{ik}) = f(f(r_{ik}, 1 - r_{ik}), r_{ij})$$
$$= f(r_{ik}, f(1 - r_{ik}, r_{ik})) = f(r_{ik}, 0.5) \qquad \forall i, k.$$

On the other hand, (1) and Proposition 1 imply

$$r_{ik} = f(r_{ij}, r_{jk}) = f(r_{ij}, f(r_{ji}, r_{ik})) = f(f(r_{ij}, r_{ji}), r_{ik})$$
$$= f(0.5, r_{ik}) \qquad \forall i, k.$$

Therefore, 0.5 is the identity element of function f.

Proposition 3 (Identity Element): Let $R = (r_{ij})$ be a reciprocal preference relation. A function $f : [0,1] \times [0,1] \to [0,1]$ verifying $r_{ik} = f(r_{ij}, r_{jk}) \ (\forall i, j, k)$ has 0.5 as its identity element

$$f(0.5, x) = f(x, 0.5) = x \qquad \forall x \in [0,1].$$

The preference value r_{ik} should not decrease when any of the preference values r_{ij}, r_{jk} increases while the other remains fixed. We impose this monotonicity (increasing) property to function f.

Property 1 (Monotonicity): Let $R = (r_{ij})$ be a reciprocal preference relation. A function $f : [0,1] \times [0,1] \to [0,1]$ verifying $r_{ik} = f(r_{ij}, r_{jk}) \ (\forall i, j, k)$ is assumed to be monotonic (increasing), i.e.

$$f(x,y) \geq f(x', y') \qquad \text{if } x \geq x' \quad \text{and} \quad y \geq y'.$$

The following result states that functional consistency with function f being monotonic, associative, self-dual, and with identity element 0.5 implies strong stochastic transitivity, and therefore, minimum transitivity as well.

Proposition 4: Let $f : [0,1] \times [0,1] \to [0,1]$ be an increasing, associative, self-dual function with identity element 0.5. Then

$$\begin{cases} f(x,y) \geq \max\{x,y\}, & \text{if } \min\{x,y\} \geq 0.5 \\ f(x,y) \leq \min\{x,y\}, & \text{if } \max\{x,y\} \leq 0.5 \\ \min\{x,y\} \leq f(x,y) \leq \max\{x,y\}, & \text{otherwise.} \end{cases}$$

The following result is obtained:

Corollary 1: Let $f : [0,1] \times [0,1] \to [0,1]$ be an increasing, associative, self-dual function with identity element 0.5. Then

1) $x \geq 0.5 \Rightarrow f(x,1) = f(1,x) = 1$.
   In particular, $f(1,1) = 1$.
2) $x \leq 0.5 \Rightarrow f(x,0) = f(0,x) = 0$.
   In particular, $f(0,0) = 0$.

Monotonicity property means that self-duality of f is not applicable when $(x,y) \in \{(0,1),(0,1)\}$. Indeed, if this were the case, then $\forall x \geq 0.5$, and we would have

$$x = f(0.5,x) = f(f(0,1),x) = f(0,f(1,x)) = f(0,1) = 0.5.$$

The same conclusion is obtained when $x \leq 0.5$. Therefore, it cannot be $f(0,1) = f(1,0) = 0.5$. So, if $f(0,1)$ ($f(1,0)$) exists, then we have the following two cases.

1) If $f(0,1) > 0.5$, then

$$f(0,1) = f(0,f(1,1)) = f(f(0,1),1) = 1.$$

2) If $f(0,1) < 0.5$, then $f(0,1) = 0$.

Proposition 5: Let $R = (r_{ij})$ be a reciprocal preference relation. If a monotonic (increasing) function $f : [0,1] \times [0,1] \to [0,1]$ verifies $r_{ik} = f(r_{ij}, r_{jk})$ ($\forall i,j,k$), then it is self-dual on $[0,1] \times [0,1]\backslash\{(0,1),(1,0)\}$ and $f(0,1), f(1,0) \in \{0,1\}$.

Another desirable property to be verified by function f should be that of continuity, because it is expected that a slight change of the values in (r_{ij}, r_{jk}) should produce a slight change in the value r_{ik}. However, continuity is not possible to be achieved at $(0,1)$ nor at $(1,0)$, because

$$\lim_{x \to 0} f(x, 1-x) \neq f(0,1) \ \wedge \ \lim_{x \to 0} f(1-x, x) \neq f(1,0).$$

Property 2 (Continuity): Let $R = (r_{ij})$ be a reciprocal preference relation. A function $f : [0,1] \times [0,1] \to [0,1]$ verifying $r_{ik} = f(r_{ij}, r_{jk})$ ($\forall i,j,k$) is assumed to be continuous on $[0,1] \times [0,1]\backslash\{(0,1),(1,0)\}$.

To conclude this section, we note that if there exist alternatives x_j, x_k, and x_l such that

$$f(r_{ij}, r_{jk}) = f(r_{ij}, r_{jl}) \qquad \forall i$$

then

$$r_{jk} = f(0.5, r_{jk}) = f(f(r_{ji}, r_{ij}), r_{jk}) = f(r_{ji}, f(r_{ij}, r_{jk}))$$
$$= f(r_{ji}, f(r_{ij}, r_{jl})) = f(f(r_{ji}, r_{ij}), r_{jl}) = f(0.5, r_{jl})$$
$$= r_{jl}.$$

Obviously, when $f(r_{kj}, r_{ji}) = f(r_{lj}, r_{ji}) \ \forall i$, we derive $r_{kj} = r_{lj}$. This property is known with the name of "cancellative" property. Due to the problems with the definition of function f when $(x,y) \in \{(0,1),(1,0)\}$, we have the following proposition.

Proposition 6 (Cancellative): Let $R = (r_{ij})$ be a reciprocal preference relation. If a monotonic (increasing) function $f : [0,1] \times [0,1] \to [0,1]$ verifies $r_{ik} = f(r_{ij}, r_{jk})$ ($\forall i,j,k$), then f is cancellative on $[0,1] \times [0,1]\backslash\{(0,1),(1,0)\}$.

Summarizing, a reciprocal preference relation $R = (r_{ij})$ is consistent if and only if

$$r_{ik} = f(r_{ij}, r_{jk}) \qquad \forall i,j,k$$

with $f : [0,1] \times [0,1] \to [0,1]$ being a continuous, monotonic increasing, associative, cancellative, self-dual function on $[0,1] \times [0,1]\backslash\{(0,1),(1,0)\}$, with identity element 0.5, and $f(0,1), f(1,0) \in \{0,1\}$.

IV. REPRESENTABLE UNINORMS AND CARDINAL CONSISTENCY OF RECIPROCAL PREFERENCE RELATIONS: THE MULTIPLICATIVE TRANSITIVITY PROPERTY

Uninorms were introduced by Yager and Rybalov [57] as a generalization of the *t*-norm and *t*-conorm, and share with them the commutativity, associativity, and monotonicity properties. It is the boundary condition or identity element that is used to generalize *t*-norms and *t*-conorms. The identity element of *t*-norms is the number 1, while for *t*-conorms, it is 0. Uninorms can have an identity element lying anywhere in the unit interval $[0,1]$.

Function f in the previous section shares all properties of a uninorm but commutativity, which cannot be directly derived from the aforementioned set of properties. However, commutativity of f can be derived indirectly from associative, cancellative, and continuity properties of f, as proved by the following result by Aczél [1].

Theorem 1: Let I be a (closed, open, half-open, finite, or infinite) proper interval of real numbers. Then $F : I^2 \longrightarrow I$ is a continuous operation on I^2 that satisfies the associativity equation

$$F(F(x,y),z) = F(x,F(y,z)) \qquad \forall x,y,z \in I$$

and is cancellative, i.e.

$$F(x_1,y) = F(x_2,y) \qquad \text{or} \qquad F(y,x_1) = F(y,x_2)$$
$$\text{implies } x_1 = x_2 \text{ for any } z \in I$$

if and only if there exists a continuous and strictly monotonic function $\phi : I \longrightarrow J$ such that

$$F(x,y) = \phi^{-1}[\phi(x) + \phi(y)] \qquad \forall x,y \in I. \qquad (2)$$

Here, J is one of the real intervals

$$]-\infty,\gamma], \quad]-\infty,\gamma[, \quad [\delta,\infty[, \quad]\delta,\infty[, \quad \text{or} \quad]-\infty,\infty[$$

for some $\gamma \leq 0 \leq \delta$. Accordingly, I has to be open at least from one side. The function in (2) is unique up to a linear transformation [$\phi(x)$ may be replaced by $(1/C)\phi(x)$, $C \neq 0$, but by no other function].

Remark 2: Although function F in Theorem 1 was not assumed to be commutative, the result (2) shows that it is. Strict monotonicity of function ϕ implies that function F is also strictly monotonic.

The representation of function F given by (2) coincides with Fodor, Yager, and Rybalov representation theorem for almost continuous uninorms U, i.e., uninorms with identity element in $]0,1[$ continuous on $[0,1] \times [0,1]\backslash\{(0,1),(1,0)\}$ [27]. Therefore, the assumption of modeling consistency of reciprocal preferences in $[0,1]$ using the functional expression (1) has solution f a representable uninorm operator with strong negator $N(x) = 1 - x$ [57]. The representation theorem also provides a relationship between the generator function ϕ and

the strong negation N: $\phi^{-1}(-\phi(x)) = N(x)$. In our case, $N(x) = 1 - x$, and we have $\phi(x) + \phi(1-x) = 0$, and in particular, $\phi(0.5) = 0$. Conforming to this result, we propose the following definition of consistent reciprocal preference relation.

Definition 3 (U-consistent reciprocal preference relation): Let U be a representable uninorm operator with strong negator $N(x) = 1 - x$. A reciprocal preference relation R on a finite set of alternatives is consistent with respect to U (U-consistent) if and only if

$$\forall i, j, k : (r_{ik}, r_{kj}) \notin \{(0,1), (1,0)\} \Rightarrow r_{ij} = U(r_{ik}, r_{kj}).$$

Tanino's multiplicative transitivity property is the U-consistency property with U being the AND-like representable cross ratio uninorm with generator function $\phi(x) = \ln(x/(1-x))$ [38]

$$U(x,y) = \begin{cases} 0, & (x,y) \in \{(0,1),(1,0)\} \\ \dfrac{xy}{xy + (1-x)(1-y)}, & \text{otherwise.} \end{cases} \tag{3}$$

This particular uninorm is the one used in the PROSPECTOR expert system [11]. Because the generator function of Theorem 1 is unique up to a linear transformation, then any two representable uninorms are order-isomorphic. Therefore, the multiplicative transitivity property is, among the many proposed properties, being characterized as the most appropriate one to model the cardinal consistency of reciprocal preference relations.

V. CONSTRUCTION OF U-CONSISTENT RECIPROCAL PREFERENCE RELATIONS

A consequence of Aczél's result is that the interval I is open at least from one side, which, in our case, means that $I \in \{]0,1], [0,1[,]0,1[\}$. If we exclude one of the extreme values of the unit interval, the other extreme should also be excluded due to the reciprocity of preferences. This implies that we should consider the functional equation (1) only when $0 < r_{ij} < 1 \ \forall i,j$. Under this restriction, in the following, we prove that for a reciprocal preference relation R

$$r_{ik} = U(r_{ij}, r_{jk}) \qquad \forall i, j, k$$

is equivalent to

$$r_{ik} = U(r_{i(i+1)}, r_{(i+1)(i+2)}, \ldots, r_{(k-2)(k-1)}, r_{(k-1)k}) \quad \forall i < k$$

where U is a representable uninorm operator with strong negator $N(x) = 1 - x$.

Proposition 7: For a reciprocal preference relation R and a representable uninorm operator with strong negator $N(x) = 1 - x$, $U:]0,1[\times]0,1[\longrightarrow]0,1[$, the following statements are equivalent
1) $r_{ik} = U(r_{ij}, r_{jk}) \ \forall i, j, k$.
2) $r_{ik} = U(r_{ij}, r_{jk}) \ \forall i \neq j \neq k$.

Proof: On the one hand, because $U(x, 1-x) = 0.5$ and reciprocity $r_{ii} = 0.5$, we have that $r_{ik} = U(r_{ij}, r_{jk})$ needs to be checked just for $i \neq k$. On the other hand, the property $U(0.5, x) = U(x, 0.5) = x$ implies that we need to consider values $j \neq i, k$. ∎

Proposition 8: For a reciprocal preference relation R and a representable uninorm operator with strong negator $N(x) = 1 - x$, $U:]0,1[\times]0,1[\longrightarrow]0,1[$, the following statements are equivalent.
1) $r_{ik} = U(r_{ij}, r_{jk}) \ \forall i \neq j \neq k$.
2) $r_{ik} = U(r_{ij}, r_{jk}) \ \forall i \neq j \neq k \wedge i < k$.

Proof: We prove 2) $\Rightarrow$ 1). If $i > k$, then

$$\begin{aligned} r_{ik} &= 1 - r_{ki} = 1 - U(r_{kj}, r_{ji}) \\ &= 1 - U(1 - r_{jk}, 1 - r_{ij}) \\ &= U(r_{jk}, r_{ij}) \\ &= U(r_{ij}, r_{jk}). \end{aligned}$$
∎

Proposition 9: For a reciprocal preference relation R and a representable uninorm operator with strong negator $N(x) = 1 - x$, $U:]0,1[\times]0,1[\longrightarrow]0,1[$, the following statements are equivalent.
1) $r_{ik} = U(r_{ij}, r_{jk}) \ \forall i \neq j \neq k \wedge i < k$.
2) $r_{ik} = U(r_{ij}, r_{jk}) \ \forall i < j < k$.

Proof: We prove 2) $\Rightarrow$ 1).
I) If $j < i < k$, then

$$\begin{aligned} U(r_{ij}, r_{jk}) &= U(r_{ij}, U(r_{ji}, r_{ik})) \\ &= U(U(r_{ij}, r_{ji}), r_{ik}) \\ &= U(0.5, r_{ik}) = r_{ik}. \end{aligned}$$

II) If $i < k < j$, then

$$\begin{aligned} U(r_{ij}, r_{jk}) &= U(U(r_{ik}, r_{kj}), r_{jk}) \\ &= U(r_{ik}, U(r_{kj}, r_{jk})) \\ &= U(r_{ik}, 0.5) = r_{ik}. \end{aligned}$$
∎

Proposition 10: For a reciprocal preference relation R and a representable uninorm operator with strong negator $N(x) = 1 - x$, $U:]0,1[\times]0,1[\longrightarrow]0,1[$, the following statements are equivalent.
1) $r_{ik} = U(r_{ij}, r_{jk}) \ \forall i < j < k$.
2) $r_{ik} = U(r_{i(i+1)}, r_{(i+1)(i+2)}, \ldots, r_{(k-2)(k-1)}, r_{(k-1)k})$ $\forall i < k$.

Proof:
1) $\Rightarrow$ 2). Let $i < k$, we have that

$$r_{ik} = U(r_{i(i+1)}, r_{(i+1)k})$$
$$r_{(i+1)k} = U(r_{(i+1)(i+2)}, r_{(i+2)k})$$
$$\vdots$$
$$r_{(k-2)k} = U(r_{(k-2)(k-1)}, r_{(k-1)k}).$$

Putting all these expressions together, and applying associativity of U, we have

$$r_{ik} = U(r_{i(i+1)}, r_{(i+1)(i+2)}, \ldots, r_{(k-2)(k-1)}, r_{(k-1)k})$$
$$\forall i < k$$

2) $\Rightarrow$ 1). Let $i < j < k$, we have that

$$U(r_{ij}, r_{jk}) = U(U(r_{i(i+1)}, r_{(i+1)(i+2)}, \ldots, r_{(j-1)j}),$$
$$U(r_{j(j+1)}, r_{(j+1)(j+2)}, \ldots, r_{(k-1)k})).$$

Associativity of U implies that

$$U(r_{ij}, r_{jk}) = U(r_{i(i+1)}, r_{(i+1)(i+2)}, \ldots, r_{(j-1)j},$$
$$r_{j(j+1)}, r_{(j+1)(j+2)}, \ldots, r_{(k-1)k}) = r_{ik}. \quad \blacksquare$$

Summarizing, we have proved the following result.

Theorem 2: For a reciprocal preference relation R and a representable uninorm operator with strong negator $N(x) = 1 - x$, $U:]0,1[\times]0,1[\longrightarrow]0,1[$, the following statements are equivalent.

1) $r_{ik} = U(r_{ij}, r_{jk})$ $\forall i, j, k$.
2) $r_{ik} = U(r_{i(i+1)}, r_{(i+1)(i+2)}, \ldots, r_{(k-2)(k-1)}, r_{(k-1)k})$ $\forall i < k$.

Given a representable uninorm operator with strong negator $N(x) = 1 - x$, U, Theorem 2 allows us to construct a U-consistent reciprocal preference relation from the following set of $n - 1$ preference values $R_I = \{r_{i(i+1)} \in]0,1[\,|\, i = 1, \ldots, n-1\}$ as follows.

1) $\forall(i, j)$ such that $j > i + 1$

$$r_{ij} = U(r_{i(i+1)}, r_{(i+1)(i+2)}, \ldots, r_{(j-1)j}).$$

2) $\forall(i, j)$ such that $j < i$

$$r_{ij} = 1 - r_{ji}.$$

For the cross ratio uninorm

$$U(x, y) = \frac{xy}{xy + (1-x)(1-y)} \qquad \forall x, y \in]0,1[$$

we have

$\forall(i, j)$ such that $j > i + 1$

$$r_{ij} = \frac{\prod_{l=0}^{j-(i+1)} r_{(i+l)(i+l+1)}}{\prod_{l=0}^{j-(i+1)} r_{(i+l)(i+l+1)} + \prod_{l=0}^{j-(i+1)} (1 - r_{(i+l)(i+l+1)})}.$$

Example 1: Suppose we have a set of four alternatives $\{x_1, x_2, x_3, x_4\}$ and certain knowledge to assure that alternative x_1 is weakly more important than alternative x_2, alternative x_2 is more important than x_3, and finally, alternative x_3 is strongly more important than alternative x_4. Suppose this situation is modeled by the preference values $\{r_{12} = 0.55, r_{23} = 0.65, r_{34} = 0.75\}$. Applying Theorem 2 with the cross ratio uninorm, we obtain the following values (with two decimal places):

$$r_{13} = \frac{r_{12} \cdot r_{23}}{r_{12} \cdot r_{23} + (1 - r_{12}) \cdot (1 - r_{23})} = 0.69$$

$$r_{14} = \frac{r_{12} \cdot r_{23} \cdot r_{34}}{r_{12} \cdot r_{23} \cdot r_{34} + (1 - r_{12}) \cdot (1 - r_{23}) \cdot (1 - r_{34})} = 0.87$$

$$r_{24} = \frac{r_{23} \cdot r_{34}}{r_{23} \cdot r_{34} + (1 - r_{23}) \cdot (1 - r_{34})} = 0.85$$

$$r_{21} = 1 - r_{12} = 0.45$$

$$r_{32} = 1 - r_{23} = 0.35$$

$$r_{43} = 1 - r_{34} = 0.25$$

$$r_{31} = 1 - r_{13} = 0.31$$

$$r_{41} = 1 - r_{14} = 0.13$$

$$r_{42} = 1 - r_{24} = 0.15$$

and therefore

$$R = \begin{pmatrix} 0.5 & 0.55 & 0.69 & 0.87 \\ 0.45 & 0.5 & 0.65 & 0.85 \\ 0.31 & 0.35 & 0.5 & 0.75 \\ 0.13 & 0.15 & 0.25 & 0.5 \end{pmatrix}$$

is a (multiplicative) consistent preference relation.

VI. Conclusion

Rationality is related to consistency, which is associated with the *transitivity property*. For reciprocal preference relations, many properties have been suggested to model transitivity, some of which have been proved to be inappropriate. Recently, a general framework for studying the transitivity of reciprocal preference relations, the cycle transitivity, was presented in [14]. Stochastic (weak, moderate, and strong) transitivity properties and product rule (multiplicative transitivity), which are properties specifically devised for probabilistic binary preference relations, have usually been proposed to model transitivity of reciprocal preference relation in fuzzy set theory. All these transitivity properties are special cases of cycle transitivity. In practical cases, any of these properties could be used to model, and therefore, to measure the consistency of reciprocal preference relations.

In this paper, we have argued that the assumption of experts being able to quantify their preferences in the domain [0,1] instead of $\{0, 1\}$ underlies unlimited computational abilities and resources from the experts. As a consequence, we have proposed to model the cardinal consistency of reciprocal preference relations via a functional equation, and we have shown that when such a function is almost continuous and monotonic (increasing), then it must be a representable uninorm. Cardinal consistency with the conjunctive representable cross ratio uninorm is equivalent to Tanino's multiplicative transitivity property. Because any two representable uninorms are order-isomorphic, multiplicative transitivity is being characterized as the most appropriate to model consistency of reciprocal preference relations. Finally, we also provided results toward the construction of consistent reciprocal preference relation from a minimum set of $(n - 1)$ preference values.

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Francisco Chiclana received the B.Sc. and Ph.D. degrees in mathematics from the University of Granada, Granada, Spain, in 1989 and 2000, respectively.

In August 2003, he joined the Centre for Computational Intelligence Research Group, De Montfort University, Leicester, U.K., where, since August 2006, he has been a Principal Lecturer and is currently a Reader of computational intelligence. He has authored or coauthored several papers published in international journals such as the IEEE TRANSACTIONS ON FUZZY SYSTEMS, the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS (PART A/PART B), the *European Journal of Operational Research*, *Fuzzy Sets and Systems*, the *International Journal of Intelligent Systems*, *Information Sciences*, and the *International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems*. He is a member of the Editorial Board of *The Open Cybernetics and Systemics Journal*. His current research interests include fuzzy preference modeling, decision-making problems with heterogeneous fuzzy information, decision support systems, the consensus reaching process, recommender systems, social networks, modeling situations with missing/incomplete information, and rationality/consistency and aggregation of information.

Enrique Herrera-Viedma was born in 1969. He received the M.Sc. and Ph.D. degrees in computer sciences from the University of Granada, Granada, Spain, in 1993 and 1996, respectively.

He is currently a Professor in the Department of Computer Science and Artificial Intelligence, University of Granada. He has authored or coauthored more than 50 papers in international journals. He has also coedited one international book and 11 special issues in international journals on topics such as computing with words and preference modeling, soft computing in information retrieval, and aggregation operators. He is a member of the Editorial Board of the journals *Fuzzy Sets and Systems*, *Soft Computing*, and the *International Journal of Information Technology and Decision Making*. His current research interests include group decision making, decision support systems, consensus models, linguistic modeling, aggregation of information, information retrieval, genetic algorithms, digital libraries, Web quality evaluation, and recommender systems.

Sergio Alonso was born in Granada, Spain, in 1979. He received the M.Sc. and Ph.D. degrees in computer sciences from the University of Granada, Granada, in 2003 and 2006, respectively.

In 2005, he received a grant by the Spanish Department for Education and Science to conduct research for a three-month period with the Centre for Computational Intelligence Research Group, De Montfort University, Leicester, U.K. He is currently a Teaching Fellow in the Department of Software Engineering, University of Granada. His current research interests include fuzzy preference modeling, modeling missing information situations, consistency properties of preferences, group decision making, decision support systems, consensus reaching processes, aggregation of information, Web quality evaluation, and recommender systems.

Francisco Herrera received the M.Sc. and Ph.D. degrees in mathematics from the University of Granada, Granada, Spain, in 1988 and 1991, respectively.

He is currently a Professor in the Department of Computer Science and Artificial Intelligence, University of Granada. He has authored or coauthored more than 140 papers in international journals. He is a coauthor of the book *Genetic Fuzzy Systems: Evolutionary Tuning and Learning of Fuzzy Knowledge Bases* (World Scientific, 2001). He has also coedited four international books and 17 special issues in international journals on different topics of soft computing. His current research interests include computing with words and decision making, data mining, data preparation, fuzzy-rule-based systems, genetic fuzzy systems, knowledge extraction based on evolutionary algorithms, memetic algorithms, and genetic algorithm. He is an Associate Editor of the journals: IEEE TRANSACTIONS ON FUZZY SYSTEMS, *Mathware and Soft Computing*, *Advances in Fuzzy Systems*, and *Advances in Computational Sciences and Technology*. He is also an Area Editor of the *Journal of Soft Computing* (area of genetic algorithms and genetic fuzzy systems), and a member of the Editorial Board of the journals *Fuzzy Sets and Systems*, *Applied Intelligence*, *Information Fusion*, *Evolutionary Intelligence*, the *International Journal of Hybrid Intelligent Systems*, *Memetic Computation*, the *International Journal of Computational Intelligence Research*, *The Open Cybernetics and Systemics Journal*, *Recent Patents on Computer Science*, *Mediterranean Journal of Artificial Intelligence*, the *International Journal of Information Technology and Intelligent Computing*.