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The standard completeness of interval-valued monoidal t-norm based logic

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ABSTRACT

In this paper we prove the strong standard completeness of interval-valued monoidal t-norm based logic (IVMTL) and some of its extensions. For other extensions we show that they are not strong standard complete. We also give a local deduction theorem for IVMTL and other extensions of interval-valued monoidal logic. Similar results are obtained for interval-valued fuzzy logics expanded with Baaz's Delta.

1. Introduction

In [32], we introduced interval-valued monoidal logic[2] (IVML). As its name suggests, the intended semantics of this logic are algebras of intervals. The idea behind interval-valued truth degrees is that they provide a way to express incomplete as well as graded knowledge (see e.g. [5,9,15,25,26,31,32]). In fact, interval-valued fuzzy sets and truth degrees are a special case of type-2 fuzzy sets and Z-numbers, which were introduced in [35,36]. It was proven in [32] that IVML is sound and complete w.r.t. triangle algebras, and that triangle algebras are equivalent with IVRLs (which are residuated lattices that have intervals as elements; the precise definition is in Definitions 4 and 5). These intervals can be taken in any residuated lattice. Residuated lattices form the semantics of Höhle's monoidal logic (ML) [20], which explains the second part of the name IVML. Numerous axiomatic extensions of IVML can be defined. All of them are sound and complete w.r.t. the corresponding subvarieties of the variety of triangle algebras. An interesting example is interval-valued monoidal t-norm based logic[3] (IVMTL), because it was proven in [34] that this logic (and its extensions) is pseudo-chain complete. This means that the semantics can be restricted to IVRLs in which the exact intervals form a chain. This is the analogue of the chain completeness of Esteva and Godo's MTL [11]. Jenei and Montagna have proven that MTL is not only chain complete, but also standard complete [23]. In the present paper, we will show that also IVMTL (and some of its extensions) is standard complete. Moreover, we will prove a local deduction theorem that holds for IVML and its extensions.

In Section 2 we recall the basic definitions and properties of fuzzy logics and their interval-valued counterparts. In Section 3 we introduce a number of specific interval-valued logics, corresponding to the commonly used (non-IV) fuzzy logics.

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[2] In [32–34] we called this logic triangle logic (TL). We decided to rename it because in our opinion the new name is more suitable, as it better describes what the logic is meant for. Moreover, using this new name allows us to name extensions of the logic in a uniform and consistent way.
[3] In [34] we called this logic pseudo-linear triangle logic (PTL). For the same reasons as for triangle logic, we decided to rename it.

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And in Section 4 we investigate which of these logics are standard complete, and which not. Moreover, we prove a local deduction theorem. In Section 5 we prove similar results for a specific expansion of IVMTL, IVMTL$_\Delta$ (and its expansions).

2. Preliminaries

IVML is basically monoidal logic (ML) [20] enriched with two unary connectives $\Box$ and $\Diamond$ (representing 'necessity' and 'possibility') and a constant $\bar{u}$ (representing 'uncertainty'). So the language of IVML consists of countably many propositional variables ($p_1, p_2, \ldots$), the constants $\bar{0}$ and $\bar{u}$, the unary operators $\Box$, $\Diamond$, the binary operators $\wedge$, $\vee$, &, $\rightarrow$, and finally the auxiliary symbols '(' and ')'. IVML-formulas are defined inductively: propositional variables, $\bar{0}$ and $\bar{u}$ are IVML-formulas; if ϕ and ψ are IVML-formulas, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \,\&\, \psi)$, $(\phi \rightarrow \psi)$, $(\Box\phi)$ and $(\Diamond\phi)$.

Remark that the set of ML-formulas is contained in the set of IVML-formulas.

The following notations are used: $\bar{1}$ for $\bar{0} \rightarrow \bar{0}$, $\neg\phi$ for $\phi \rightarrow \bar{0}$, ϕ^2 for $\phi \,\&\, \phi$, ϕ^n (with $n = 3, 4, 5, \ldots$) for $(\phi^{n-1}) \,\&\, \phi$ (moreover, ϕ^0 is $\bar{1}$ and ϕ^1 is ϕ), and $\phi \leftrightarrow \psi$ for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$, for formulas ϕ and ψ.

The axioms[4] of IVML are those of ML, i.e.,

(ML.1) $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$,
(ML.2) $\phi \rightarrow (\phi \vee \psi)$,
(ML.3) $\psi \rightarrow (\phi \vee \psi)$,
(ML.4) $(\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\phi \vee \psi) \rightarrow \chi))$,
(ML.5) $(\phi \wedge \psi) \rightarrow \phi$,
(ML.6) $(\phi \wedge \psi) \rightarrow \psi$,
(ML.7) $(\phi \,\&\, \psi) \rightarrow \phi$,
(ML.8) $(\phi \,\&\, \psi) \rightarrow (\psi \,\&\, \phi)$,
(ML.9) $(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \wedge \chi)))$,
(ML.10) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \,\&\, \psi) \rightarrow \chi)$,
(ML.11) $((\phi \,\&\, \psi) \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi))$,
(ML.12) $\bar{0} \rightarrow \phi$,

complemented with

(IVML.1) $\Box\phi \rightarrow \phi$, (IVML.1′) $\phi \rightarrow \Diamond\phi$,
(IVML.2) $\Box\phi \rightarrow \Box\Box\phi$, (IVML.2′) $\Diamond\Diamond\phi \rightarrow \Diamond\phi$,
(IVML.3) $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$, (IVML.3′) $(\Diamond\phi \wedge \Diamond\psi) \rightarrow \Diamond(\phi \wedge \psi)$,
(IVML.4) $\Box(\phi \vee \psi) \rightarrow (\Box\phi \vee \Box\psi)$, (IVML.4′) $\Diamond(\phi \vee \psi) \rightarrow (\Diamond\phi \vee \Diamond\psi)$,
(IVML.5) $\neg\Box\bar{u}$, (IVML.5′) $\Diamond\bar{u}$,
(IVML.6) $\Diamond\phi \rightarrow \Box\Diamond\phi$, (IVML.6′) $\Diamond\Box\phi \rightarrow \Box\phi$,
(IVML.7) $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$,
(IVML.8) $(\Box\phi \leftrightarrow \Box\psi) \,\&\, (\Diamond\phi \leftrightarrow \Diamond\psi) \rightarrow (\phi \leftrightarrow \psi)$,
(IVML.9) $(\Box\phi \rightarrow \Box\psi) \rightarrow \Box(\Box\phi \rightarrow \Box\psi)$.

The deduction rules are modus ponens (MP, from ϕ and $\phi \rightarrow \psi$ infer ψ), $\Box$-necessitation[5] (G, from ϕ infer $\Box\phi$) and monotonicity of $\Diamond$ (M$\Diamond$, from $\phi \rightarrow \psi$ infer $\Diamond\phi \rightarrow \Diamond\psi$). Proofs and the provability relation $\vdash_{IVML}$ are defined in the usual way.

IVML is a logic which has interval-valued structures as its (general) semantics (hence its name). To see this, we recall the following definitions and results from [32].

ML is sound and complete w.r.t. the variety of residuated lattices[6] [7], which are structures $\mathcal{L} = (L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ in which $\sqcap$, $\sqcup$, $*$ and $\Rightarrow$ are binary operators on the set L and

- $(L, \sqcap, \sqcup)$ is a bounded lattice with 0 as smallest and 1 as greatest element,
- $*$ is commutative and associative, with 1 as neutral element, and
- $x * y \leqslant z$ iff $x \leqslant y \Rightarrow z$ for all x, y and z in L (residuation principle).

[4] Some of these axioms are referred to by a specific name. In [20], ML.1 is called 'syllogism law', while Hájek uses 'transitivity of implication' in [18]. Other names in [18] are 'commutativity of &-conjunction' for ML.8, 'ex falso quodlibet' for ML.12 and 'residuation' for the combination of ML.10 and ML.11 (which are called 'importation law' and 'exportation law' in [20]).

[5] In [32,34], $\Box$-necessitation was called generalization.

[6] In the literature (e.g. in [20]), the name residuated lattice is sometimes used for more general structures than what we call residuated lattices. In the most general terminology, our structures would be called bounded integral commutative residuated lattices.

ML is the basis for a number of well-known stronger formal fuzzy logics, such as Esteva and Godo's monoidal t-norm based logic (MTL) [11], Hájek's basic logic (BL) [18], Łukasiewicz logic (LL) [24], intuitionistic logic (IL) [19] and Gödel logic (GL) [8,16]. These logics are sound and complete w.r.t. MTL-algebras, BL-algebras, MV-algebras (or, equivalently, Wajsberg algebras [14]), Heyting-algebras and G-algebras, respectively. Below, we recall the definitions of these concepts, along with some other important notions. We refer to [4,12,17] for a comprehensive overview of these and other logics.

All these extensions of ML satisfy the following local deduction theorem:

Proposition 1. *Let $\Gamma \cup \{\phi, \psi\}$ be a set of ML-formulas, and* **L** *be an extension of ML.*

Then the following are equivalent:

- $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$.
- *There is an integer n such that* $\Gamma \vdash_{\mathbf{L}} \phi^n \to \psi$.

ML and its axiomatic extensions can be expanded with a unary connective Δ, called Baaz's Delta [1]. The formulas of these logics will be called ML_Δ-formulas. The logic ML_Δ is defined as ML extended with the following axioms[7] and deduction rule for Δ:

$$
\begin{aligned}
&(\Delta 1) \quad \Delta\phi \vee \neg\Delta\phi,\\
&(\Delta 2) \quad \Delta(\phi \vee \psi) \to (\Delta\phi \vee \Delta\psi),\\
&(\Delta 3) \quad \Delta\phi \to \phi,\\
&(\Delta 5) \quad \Delta(\phi \to \psi) \to (\Delta\phi \to \Delta\psi),
\end{aligned}
$$

and Δ-necessitation (N, from ϕ infer $\Delta\phi$).

For ML_Δ and its extensions, we have the following deduction theorem.

Proposition 2. *Let $\Gamma \cup \{\phi, \psi\}$ be a set of ML_Δ-formulas, and* **L** *be an extension of ML_Δ.*

Then the following are equivalent:

- $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$,
- $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \psi$.

Axiomatic extensions of MTL (which is ML extended with the axiom $(\phi \to \psi) \vee (\psi \to \phi)$) are specific kinds of core fuzzy logics (see [4] for more details). Axiomatic extensions of MTL_Δ are specific kinds of Δ-core fuzzy logics. In core fuzzy logics and Δ-core fuzzy logics, the language is allowed to have more connectives than the ones we use in this paper (but at most a countable amount).

Definition 3. We will use the notations $\neg x$ for $x \Rightarrow 0$, $x \Longleftrightarrow y$ for $(x \Rightarrow y) \sqcap (y \Rightarrow x)$ and x^n for $\underbrace{x * x * \cdots * x}_{n \text{ times}}$. Moreover, we assume $x^0 = 1$.

- An MTL-algebra [11] is a prelinear residuated lattice, i.e., a residuated lattice in which $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$ for all x and y in L.
- A BL-algebra [18] is a divisible MTL-algebra, i.e., an MTL-algebra in which $x \sqcap y = x * (x \Rightarrow y)$ for all x and y in L. The weaker property $x \sqcap y = (x * (x \Rightarrow y)) \sqcup (y * (y \Rightarrow x))$ is called weak divisibility [31,32] and holds in all MTL-algebras.
- An MV-algebra [2,3] is a BL-algebra in which the negation is an involution, i.e., $(x \Rightarrow 0) \Rightarrow 0 = x$ for all x in L.
- A Heyting-algebra, or pseudo-Boolean algebra [30], is a residuated lattice in which $x * x = x$ for all x in L, or, equivalently, in which $x * y = x \sqcap y$ for all x and y in L.
- A G-algebra [18] is a prelinear Heyting-algebra.
- A Boolean algebra is an MV-algebra that is also a Heyting-algebra.

By adding a unary operator Δ satisfying $\Delta 1 = 1$, $\Delta x \sqcup \neg\Delta x = 1$, $\Delta(x \sqcup y) \leqslant \Delta x \sqcup \Delta y$, $\Delta x \leqslant x$ and $\Delta(x \Rightarrow y) \leqslant \Delta x \Rightarrow \Delta y$, for all x and y, we can define the 'Δ-companions' of these algebras (e.g. MTL_Δ-algebra, G_Δ-algebra, ...). If a residuated lattice satisfies $x \sqcup y = ((x \Rightarrow y) \Rightarrow y) \sqcap ((y \Rightarrow x) \Rightarrow x)$, for all x and y in L, then it is called $\sqcup$-definable [11,12]. The stronger property $x \sqcup y = (x \Rightarrow y) \Rightarrow y$ is called strong $\sqcup$-definability[8] [31,32]. Other interesting properties are the law of excluded middle[9] $(x \sqcup \neg x = 1)$, pseudocomplementation $(x \sqcap \neg x = 0)$, cancellation $(\neg x \sqcup ((x \Rightarrow (x * y)) \Rightarrow y) = 1)$, weak cancellation $(\neg(x * y) \sqcup ((x \Rightarrow (x * y)) \Rightarrow y) = 1)$ and weak nilpotent minimum $(\neg(x * y) \sqcup ((x \sqcap y) \Rightarrow (x * y)) = 1)$.

[7] Note that we left out (Δ4). In Section 5 we shall show that $\Delta\phi \to \Delta\Delta\phi$ (which is known as (Δ4)) is provable from ML_Δ.

[8] Strong $\sqcup$-definable residuated lattices are exactly MV-algebras [20].

[9] Residuated lattices satisfying the law of excluded middle are exactly Boolean algebras.

Definition 4. Given a lattice $\mathcal{L} = (L, \sqcap, \sqcup)$, its triangularization $\mathbb{T}(\mathcal{L})$ is the structure $\mathbb{T}(\mathcal{L}) = (Int(\mathcal{L}), \sqcap, \bigsqcup)$ defined by

- $Int(\mathcal{L}) = \{[x_1, x_2] | (x_1, x_2) \in L^2 \text{ and } x_1 \leqslant x_2\}$,
- $[x_1, x_2] \sqcap [y_1, y_2] = [x_1 \sqcap y_1, x_2 \sqcap y_2]$,
- $[x_1, x_2] \bigsqcup [y_1, y_2] = [x_1 \sqcup y_1, x_2 \sqcup y_2]$.

The set $D_{\mathcal{L}} = \{[x, x] | x \in L\}$ is called the diagonal of $\mathbb{T}(\mathcal{L})$.
In particular, the triangularization of $([0, 1], \min, \max)$ is denoted as $\mathcal{L}^I = (L^I, \sqcap, \sqcup)$.

Definition 5. An interval-valued residuated lattice (IVRL) is a residuated lattice $(Int(\mathcal{L}), \sqcap, \bigsqcup, \odot, \Rightarrow_\odot, [0, 0], [1, 1])$ on the triangularization $\mathbb{T}(\mathcal{L})$ of a bounded lattice $\mathcal{L}$, in which the diagonal $D_{\mathcal{L}}$ is closed under $\odot$ and $\Rightarrow_\odot$, i.e., $[x, x] \odot [y, y] \in D_{\mathcal{L}}$ and $[x, x] \Rightarrow_\odot [y, y] \in D_{\mathcal{L}}$ for all x, y in L.

Example 6 [6]. If T is a left-continuous t-norm on $([0, 1], \min, \max)$, $\alpha \in [0, 1]$ and the mapping $\mathcal{T}_{T,\alpha}$ is defined, for $x = [x_1, x_2]$ and $y = [y_1, y_2]$ in L^I, by the formula

$$\mathcal{T}_{T,\alpha}(x, y) = [T(x_1, y_1), \max(T(\alpha, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))], \tag{1}$$

then $(L^I, \sqcap, \sqcup, \mathcal{T}_{T,\alpha}, \mathcal{I}_{\mathcal{T}_{T,\alpha}}, [0, 0], [1, 1])$ is an IVRL, in which $\mathcal{I}_{\mathcal{T}_{T,\alpha}}$ is the residual implicator of $\mathcal{T}_{T,\alpha}$:

$$\mathcal{I}_{\mathcal{T}_{T,\alpha}}(x, y) = [\min(I_T(x_1, y_1), I_T(x_2, y_2)), \min(I_T(T(x_2, \alpha), y_2), I_T(x_1, y_2))].$$

In [32], we introduced the notion of triangle algebra, a structure that serves as an equational representation for an interval-valued residuated lattice. Triangle algebras form the link between IVRLs and IVML.

Definition 7. A triangle algebra is a structure $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, in which $(A, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, ν and μ are unary operators, u a constant, and satisfying the following conditions:

$$\begin{array}{llll}
(\mathrm{T.1}) & \nu x \leqslant x, & (\mathrm{T.1}') & x \leqslant \mu x, \\
(\mathrm{T.2}) & \nu x \leqslant \nu\nu x, & (\mathrm{T.2}') & \mu\mu x \leqslant \mu x, \\
(\mathrm{T.3}) & \nu(x \sqcap y) = \nu x \sqcap \nu y, & (\mathrm{T.3}') & \mu(x \sqcap y) = \mu x \sqcap \mu y, \\
(\mathrm{T.4}) & \nu(x \sqcup y) = \nu x \sqcup \nu y, & (\mathrm{T.4}') & \mu(x \sqcup y) = \mu x \sqcup \mu y, \\
(\mathrm{T.5}) & \nu u = 0, & (T.5') & \mu u = 1, \\
(\mathrm{T.6}) & \nu\mu x = \mu x, & (\mathrm{T.6}') & \mu\nu x = \nu x, \\
(\mathrm{T.7}) & \nu(x \Rightarrow y) \leqslant \nu x \Rightarrow \nu y, & & \\
(\mathrm{T.8}) & (\nu x \Leftrightarrow \nu y) * (\mu x \Leftrightarrow \mu y) \leqslant (x \Leftrightarrow y), & & \\
(\mathrm{T.9}) & \nu x \Rightarrow \nu y \leqslant \nu(\nu x \Rightarrow \nu y). & &
\end{array}$$

A triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u_A, 1_A)$ is called a standard triangle algebra iff $(A, \sqcap, \sqcup) = \mathcal{L}^I$.

In a standard triangle algebra $(L^I, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u_A, 1_A)$, $0_A = [0, 0]$, $1_A = [1, 1]$, $u = [0, 1]$, $\nu[x_1, x_2] = [x_1, x_1]$ and $\mu[x_1, x_2] = [x_2, x_2]$ for all $[x_1, x_2]$ in L^I. This is a consequence of Propositions 19 and 21 in [32].

In [32], we also established a one-to-one correspondence between interval-valued residuated lattices (IVRLs) and triangle algebras. The correspondence is shown in Fig. 1. The unary operators ν and μ correspond with the mappings that map $[x_1, x_2]$ to $[x_1, x_1]$ and $[x_2, x_2]$ respectively. We call these mappings in IVRLs the vertical and horizontal projection (p_ν and p_h). The constant u corresponds to $[0, 1]$. Theorem 8 gives this connection in more detail:

Theorem 8 [32]. *There is a one-to-one correspondence between the class of IVRLs and the class of triangle algebras. Every extended IVRL*[10] *is a triangle algebra and conversely, every triangle algebra is isomorphic to an extended IVRL.*

In [32], it was verified that IVML is sound and complete w.r.t. triangle algebras. Because of Theorem 8, this implies that IVML is sound and complete w.r.t. extended IVRLs. Axiomatic extensions of IVML are sound and complete w.r.t. the corresponding subclasses of the class of extended IVRLs.

Definition 9 [32]. Let $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra. An element x in A is called exact if $\nu x = x$. The set of exact elements of $\mathcal{A}$ is denoted by $E(\mathcal{A})$.

Using the isomorphism in Fig. 1, the set of exact elements of a triangle algebra corresponds to the diagonal of the isomorphic (extended) IVRL. In this paper we will sometimes use the term 'diagonal' for triangle algebras as well.

It was proven in [32] that $E(\mathcal{A})$ is closed under all the defined operations on $\mathcal{A}$. So $(E(\mathcal{A}), \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice, that we will denote as $\mathcal{E}(\mathcal{A})$. Every property in Definition 3 (prelinearity, divisibility, ...) can therefore be weakened, by imposing it on $E(\mathcal{A})$ (instead of A) only. We will denote this with the prefix 'pseudo'. For example, a triangle algebra is said

[10] An extended IVRL is simply an IVRL in which the two mentioned projections are defined and the constant $[0, 1]$ is fixed.

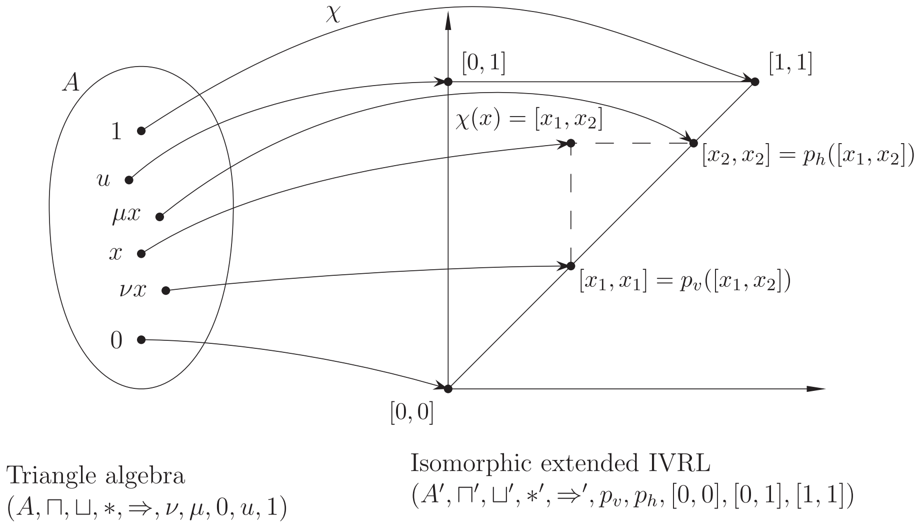

Fig. 1. The isomorphism χ from a triangle algebra to an IVRL.

to be pseudo-linear if its set of exact elements is linearly ordered (by the original (restricted) ordering). Another example: a triangle algebra is pseudo-divisible if $\nu x \sqcap \nu y = \nu x * (\nu x \Rightarrow \nu y)$ for all x and y in A ($E(\mathcal{A})$ consists exactly of the elements of the form[11] νx).

For any x in a triangle algebra, it holds that $x = \nu x \sqcup (\mu x \sqcap u)$ (see [33]). Therefore, x is completely determined by νx and μx (which are elements of $E(\mathcal{A})$): if $\nu x = \nu y$ and $\mu x = \mu y$, then $x = y$.

In [33] we proved that

Theorem 10. *In a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$, the implication $\Rightarrow$ and the product $*$ are completely determined by their action on $E(\mathcal{A})$ and the value of $\mu(u * u)$. More specifically:*

- $\nu(x \Rightarrow y) = (\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y)$,
- $\mu(x \Rightarrow y) = (\mu x \Rightarrow (\mu(u * u) \Rightarrow \mu y)) \sqcap (\nu x \Rightarrow \mu y)$,
- $\nu(x * y) = \nu x * \nu y$,
- $\mu(x * y) = (\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u))$.

Because of Theorem 10, Example 6 gives all standard triangle algebras (i.e., all IVRLs on L^I).

Proposition 11 [34]. *For any residuated lattice $\mathcal{L}$ and $\alpha \in L$, there is a triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ such that (up to isomorphism) $\mathcal{E}(\mathcal{A})$ is $\mathcal{L}$ and $\mu(u * u) = \alpha$.*

In the interval-valued setting, evaluations and models are defined in the same way as in the known fuzzy setting.

Definition 12 [32]. Let $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ be a triangle algebra, Γ a theory (i.e., a set of (IVML-)formulas). An $\mathcal{A}$-evaluation is a mapping e from the set of IVML-formulas to A that satisfies, for each two formulas ϕ and ψ: $e(\phi \wedge \psi) = e(\phi) \sqcap e(\psi)$, $e(\phi \vee \psi) = e(\phi) \sqcup e(\psi)$, $e(\phi \& \psi) = e(\phi) * e(\psi)$, $e(\phi \to \psi) = e(\phi) \Rightarrow e(\psi)$, $e(\Box\phi) = \nu e(\phi)$, $e(\Diamond\phi) = \mu e(\phi)$, $e(\bar{0}) = 0$ and $e(\bar{u}) = u$. If an $\mathcal{A}$-evaluation e satisfies $e(\chi) = 1$ for every χ in Γ, it is called an $\mathcal{A}$-model for Γ.

We write $\Gamma \models_{\mathcal{A}} \phi$ if $e(\phi) = 1$ for all $\mathcal{A}$-models e for Γ.

We conclude this section with the definition of the different kinds of completeness an axiomatic extension of IVML can enjoy. These are comparable to the different kinds of completeness for fuzzy logics (see, e.g., [4,22]).

Definition 13. Let **L** be an axiomatic extension of IVML.

An **L**-algebra is a triangle algebra that satisfies the properties corresponding to the axioms that were added to IVML in order to obtain **L**.[12]

L is called pseudo-chain complete if the following equivalence holds for all IVML-formulas ϕ: $\vdash_{\mathbf{L}} \phi$ iff $\models_{\mathcal{A}} \phi$ for all pseudo-linear **L**-algebras $\mathcal{A}$.

[11] Remark that $E(\mathcal{A})$ also consists exactly of the elements of the form μx. So pseudo-divisibility might as well be expressed by $\mu x \sqcap \mu y = \mu x * (\mu x \Rightarrow \mu y)$ or $\mu x \sqcap \nu y = \mu x * (\mu x \Rightarrow \nu y)$, for all x and y in A. And similarly for other properties (pseudo-prelinearity, pseudo-cancellation, ...), of course.

[12] For example, if **L** is IVML extended with the axiom scheme $\neg\neg\phi \to \phi$, then an **L**-algebra is a triangle algebra satisfying $\neg\neg x \Rightarrow x = 1$, in other words a triangle algebra with an involutive negation.

Table 1
Some axioms in interval-valued fuzzy logics.

Axiom	Name
$(\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi)$	Pseudo-prelinearity (PP)
$((\Box\phi \to \Box\psi) \to \Box\psi) \to (\Box\phi \vee \Box\psi)$	Pseudo-strong $\vee$-definability (PS$\vee$)
$\Box\phi \vee \neg\Box\phi$	Pseudo-law of excluded middle (PLEM)
$\Box\phi \to (\Box\phi \& \Box\phi)$	Pseudo-contraction (PCon)
$\neg(\Box\phi \& \Box\psi) \vee ((\Box\phi \wedge \Box\psi) \to (\Box\phi \& \Box\psi))$	Pseudo-weak nilpotent minimum (PWNM)
$\neg\neg\Box\phi \to \Box\phi$	Pseudo-involution (PInv)
$\neg(\Box\phi \wedge \neg\Box\phi)$	Pseudo-pseudocomplementation (PPC)
$\neg(\Box\phi \& \Box\psi) \vee ((\Box\phi \to (\Box\phi \& \Box\psi)) \to \Box\psi)$	Pseudo-weak cancellation (PWCan)
$\neg\Box\phi \vee ((\Box\phi \to (\Box\phi \& \Box\psi)) \to \Box\psi)$	Pseudo-cancellation (PCan)
$(\Box\phi \wedge \Box\psi) \to (\Box\phi \& (\Box\phi \to \Box\psi))$	Pseudo-divisibility (PDiv)

L is called finite strong pseudo-chain complete if the following equivalence holds for all finite sets $\Gamma \cup \{\phi\}$ of IVML-formulas: $\Gamma \vdash_{\mathbf{L}} \phi$ iff $\Gamma \models_{\mathcal{A}} \phi$ for all pseudo-linear **L**-algebras $\mathcal{A}$.
L is called strong pseudo-chain complete if the following equivalence holds for all sets $\Gamma \cup \{\phi\}$ of IVML-formulas: $\Gamma \vdash_{\mathbf{L}} \phi$ iff $\Gamma \models_{\mathcal{A}} \phi$ for all pseudo-linear **L**-algebras $\mathcal{A}$.
L is called standard complete if the following equivalence holds for all IVML-formulas ϕ: $\vdash_{\mathbf{L}} \phi$ iff $\models_{\mathcal{A}} \phi$ for all standard **L**-algebras $\mathcal{A}$.
L is called finite strong standard complete if the following equivalence holds for all finite sets $\Gamma \cup \{\phi\}$ of IVML-formulas: $\Gamma \vdash_{\mathbf{L}} \phi$ iff $\Gamma \models_{\mathcal{A}} \phi$ for all standard **L**-algebras $\mathcal{A}$.
L is called strong standard complete if the following equivalence holds for all sets $\Gamma \cup \{\phi\}$ of IVML-formulas: $\Gamma \vdash_{\mathbf{L}} \phi$ iff $\Gamma \models_{\mathcal{A}} \phi$ for all standard **L**-algebras $\mathcal{A}$.

3. Axiomatic extensions of IVML

Now we introduce some extensions of IVML, by adding well-known[13] axiom schemes. They are listed in Tables 1 and 2. Remark that these axiom schemes are applied to formulas of the form $\Box\phi$ and not to all formulas (as usual). As the image of a triangle algebra $(A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ under ν is the set $E(\mathcal{A})$ of exact elements,[14] this means that the axioms schemes do not hold for all truth values, but only for exact truth values. This is not a drawback. On the contrary, it is precisely what we want because the exact truth values are easier to interpret and handle. Moreover, using Theorem 10, for all axiom schemes equivalent axiom schemes can be found that only involve formulas of the form $\Box\phi$ and $\Diamond\phi$.

All these extensions of IVML are sound and (strong) complete w.r.t. their corresponding subvariety of the variety of triangle algebras [32]. For example, IVSBL is sound and complete w.r.t. the variety of triangle algebras satisfying $(\nu x \Rightarrow \nu y) \sqcup (\nu y \Rightarrow \nu x) = 1$, $\nu x \sqcap \nu y \leqslant \nu x * (\nu x \Rightarrow \nu y)$ and $(\nu x \sqcap \neg\nu x) = 0$.

Moreover, they are all[15] extensions of IVMTL and therefore all these logics are also strong complete w.r.t. their corresponding subclass of the class of pseudo-linear triangle algebras (in other words, they are strong pseudo-chain complete [34]).

For some of these logics, we can use these completeness results and use known algebraic properties of triangle algebras [34] to derive alternative defining axiom schemes. For example, IVCPC can also be defined as IVML extended with the axiom scheme $(\phi \to \psi) \vee (\psi \to \phi)$ (because a triangle algebra satisfies the pseudo-law of excluded middle iff it is prelinear); and IVBL can also be defined as IVMTL extended with the axiom scheme $(\phi \wedge \psi) \to ((\phi \& (\phi \to \psi)) \vee (\psi \& (\psi \to \phi)))$ (because a pseudo-prelinear triangle algebra is pseudo-divisible iff it is weak divisible).

In the next section we will prove that IVMTL and some of its extensions are strong standard complete and a fortiori also standard complete. For the other defined extensions we will prove that they are not strong standard complete. We will also give a local deduction theorem for all these logics.

4. Main results

In [4] it is shown that strong standard completeness of a propositional fuzzy logic is equivalent with the real-chain embedding property of that logic, and that MTL, G, WNM, IMTL, NM and SMTL satisfy this property. We will use these results in the next theorem to show that their interval-valued counterparts also satisfy strong standard completeness.

Theorem 14 (*Strong standard completeness*). *For each set of IVML-formulas $\Gamma \cup \{\phi\}$, the following four statements are equivalent:*

(1) *ϕ can be deduced from Γ in IVMTL ($\Gamma \vdash_{IVMTL} \phi$),*
(2) *for every pseudo-prelinear triangle algebra $\mathcal{A}$, $\Gamma \models_{\mathcal{A}} \phi$ (i.e., for every $\mathcal{A}$-model e of Γ, $e(\phi) = 1$),*

[13] For a more detailed overview, we refer to [4,12].
[14] Note that the image under μ is also $E(\mathcal{A})$. All axioms schemes in Table 1 can also be given in an equivalent way by changing $\Box\phi$ to $\Diamond\phi$ and/or $\Box\psi$ to $\Diamond\psi$.
[15] Indeed, also in IVŁ and IVCPC, $(\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi)$ can be proven.

Table 2
Some axiomatic extensions of IVML obtained by adding the corresponding axioms.

Logic	Additional axioms
IVMTL	(PP)
IVŁ	(PS$\vee$)
IVCPC	(PLEM)
IVG	(PP) and (PCon)
IVWNM	(PP) and (PWNM)
IVIMTL	(PP) and (PInv)
IVNM	(PP), (PWNM) and (PInv)
IVSMTL	(PP) and (PPC)
IVWCMTL	(PP) and (PWCan)
IVΠMTL	(PP) and (PCan)
IVBL	(PP) and (PDiv)
IVΠ	(PP), (PDiv) and (PCan)
IVSBL	(PP), (PPC) and (PDiv)

(3) *for every pseudo-linear triangle algebra* $\mathcal{A}$, $\Gamma \models_{\mathcal{A}} \phi$,
(4) *for every standard triangle algebra* $\mathcal{A}$, $\Gamma \models_{\mathcal{A}} \phi$.

Proof. The equivalence of the first three statements was already proven in [32,34]. We will now prove that (4) implies (3). This suffices to prove the theorem, as (3) obviously implies (4).

Suppose (3) does not hold. Thus there exists a pseudo-linear triangle algebra $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u, 1_A)$ and an $\mathcal{A}$-model e of Γ such that $e(\phi) < 1_A$. Clearly, only evaluations of subformulas of $\Gamma \cup \{\phi\}$ are relevant, therefore we can assume, without loss of generality, that $\mathcal{A}$ is at most countably generated (as the set of IVML-formulas is countable), and therefore at most countable. Because $\mathcal{E}(\mathcal{A}) = (D, \sqcap_D, \sqcup_D, *_D, \Rightarrow_D, 0_A, 1_A)$, in which $D = E(\mathcal{A})$ and $\sqcap_D$, $\sqcup_D$, $*_D$ and $\Rightarrow_D$ are the restrictions of $\sqcap$, $\sqcup$, $*$ and $\Rightarrow$ to D, is an MTL-chain (i.e., a linearly ordered MTL-algebra), we know from [23] that there exists an embedding i from $\mathcal{E}(\mathcal{A})$ into a standard MTL-algebra $([0,1], \min, \max, \circ, \Rightarrow_\circ, 0, 1)$.

Now we define a standard triangle algebra $\mathcal{A}'$ and a mapping j from $\mathcal{A}$ to $\mathcal{A}'$ in the following way: $\mathcal{A}' := (L^I, \inf, \sup, \odot, \rightsquigarrow, p_v, p_h, [0,0], [0,1], [1,1])$, with

- $\inf([x_1,x_2],[y_1,y_2]) = [\min(x_1,y_1), \min(x_2,y_2)]$,
- $\sup([x_1,x_2],[y_1,y_2]) = [\max(x_1,y_1), \max(x_2,y_2)]$,
- $[x_1,x_2] \odot [y_1,y_2] = [x_1 \circ y_1, \max(x_1 \circ y_2, x_2 \circ y_1, x_2 \circ y_2 \circ i(\mu(u * u)))]$,
- $[x_1,x_2] \rightsquigarrow [y_1,y_2] = [\min(x_1 \Rightarrow_\circ y_1, x_2 \Rightarrow_\circ y_2), \min(x_1 \Rightarrow_\circ y_2, (x_2 \circ i(\mu(u * u))) \Rightarrow_\circ y_2)]$,
- $p_v([x_1,x_2]) = [x_1,x_1]$,
- $p_h([x_1,x_2]) = [x_2,x_2]$ and
- $j(x) = [i(\nu x), i(\mu x)]$.

To verify that $\mathcal{A}'$ is indeed a standard triangle algebra, note that $(\{[x,x] | x \in [0,1]\}, \inf, \sup, \odot, \rightsquigarrow, [0,0], [1,1])$ is a subalgebra of $\mathcal{A}'$ isomorphic to $([0,1], \min, \max, \circ, \Rightarrow_\circ, 0, 1)$ and check Example 6 and Theorem 8. Now we show that j is an embedding from $\mathcal{A}$ into $\mathcal{A}'$:

$$j(u) = [i(\nu u), i(\mu u)] = [i(0_A), i(1_A)] = [0,1]$$

(and similarly for $j(0_A) = [0,0]$ and $j(1_A) = [1,1]$),

$$\begin{aligned} j(x \sqcap y) &= [i(\nu(x \sqcap y)), i(\mu(x \sqcap y))] = [i(\nu x \sqcap \nu y), i(\mu x \sqcap \mu y)] = [i(\nu x \sqcap_D \nu y), i(\mu x \sqcap_D \mu y)] \\ &= [\min(i(\nu x), i(\nu y)), \min(i(\mu x), i(\mu y))] = \inf([i(\nu x), i(\mu x)], [i(\nu y), i(\mu y)]) = \inf(j(x), j(y)) \end{aligned}$$

(and similarly for $x \sqcup y$),

$$j(\nu x) = [i(\nu\nu x), i(\mu\nu x)] = [i(\nu x), i(\nu x)] = p_v([i(\nu x), i(\mu x)]) = p_v(j(x))$$

(and similarly for μx),

$$\begin{aligned} j(x * y) &= [i(\nu(x * y)), i(\mu(x * y))] \\ &= [i(\nu x * \nu y), i((\nu x * \mu y) \sqcup (\mu x * \nu y) \sqcup (\mu x * \mu y * \mu(u * u)))] \\ &= [i(\nu x *_D \nu y), i((\nu x *_D \mu y) \sqcup_D (\mu x *_D \nu y) \sqcup_D (\mu x *_D \mu y *_D \mu(u * u)))] \\ &= [i(\nu x) \circ i(\nu y), \max(i(\nu x) \circ i(\mu y), i(\mu x) \circ i(\nu y), i(\mu x) \circ i(\mu y) \circ i(\mu(u * u)))] \\ &= [i(\nu x), i(\mu x)] \odot [i(\nu y), i(\mu y)] \\ &= j(x) \odot j(y), \end{aligned}$$

$$\begin{aligned} j(x \Rightarrow y) &= [i(\nu(x \Rightarrow y)), i(\mu(x \Rightarrow y))] \\ &= [i((\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y)), i((\nu x \Rightarrow \mu y) \sqcap ((\mu x * \mu(u * u)) \Rightarrow \mu y))] \\ &= [i((\nu x \Rightarrow_D \nu y) \sqcap_D (\mu x \Rightarrow_D \mu y)), i((\nu x \Rightarrow_D \mu y) \sqcap_D ((\mu x *_D \mu(u * u)) \Rightarrow_D \mu y))] \\ &= [\min(i(\nu x) \Rightarrow_\circ i(\nu y), i(\mu x) \Rightarrow_\circ i(\mu y)), \quad \min(i(\nu x) \Rightarrow_\circ i(\mu y), (i(\mu x) \circ i(\mu(u * u))) \Rightarrow_\circ i(\mu y))] \\ &= [i(\nu x), i(\mu x)] \rightsquigarrow [i(\nu y), i(\mu y)] \\ &= j(x) \rightsquigarrow j(y) \end{aligned}$$

and

$$\begin{aligned} j(x) = j(y) \text{ iff } & (i(\nu x) = i(\nu y) \text{ and } i(\mu x) = i(\mu y)) \\ \text{iff } & (\nu x = \nu y \text{ and } \mu x = \mu y) \\ \text{iff } & x = y. \end{aligned}$$

Now remark that e', defined by $e'(\psi) = j(e(\psi))$, is an $\mathcal{A}'$-model of Γ such that $e'(\phi) < 1$, which concludes the proof. □

This theorem can also be used, mutatis mutandis, for IVG, IVWNM, IVIMTL, IVNM and IVSMTL, because G, WNM, IMTL, NM and SMTL satisfy the real-chain embedding property, just like MTL.

Remark 15. Remark that basically what we do in the proof is applying the real-chain embedding property to the diagonal of a (countable) pseudo-linear triangle algebra, which gives us an embedding of this diagonal in a standard MTL-chain. This embedding can be extended to an embedding of the whole triangle algebra in a standard triangle algebra. This interval-valued counterpart of the real-chain embedding property might be called 'pseudo-real-chain embedding property' and enables us to prove the strong standard completeness.

- Theorem 14 does not only hold for IVMTL, IVG, IVWNM, IVIMTL, IVNM and IVSMTL, but for every interval-valued companion IVL (defined in the same way as the examples in Table 2) of a core fuzzy logic L without extra connectives that satisfies strong standard completeness (or, equivalently, the real chain embedding property). In short: if a core fuzzy logic L without extra connectives[16] is strong standard complete, then its interval-valued companion IVL is strong standard complete.
- In fact, Theorem 14 can be generalized even a bit more. Indeed, also for other kinds of strong completeness (i.e., not necessarily strong standard completeness), we have a connection between a core fuzzy logic L without extra connectives and its interval-valued companion IVL: if L is strong complete w.r.t. a class $\mathbb{K}$ of L-chains, then IVL is strong complete w.r.t. the class $TA(\mathbb{K})$ (and vice versa), with $TA(\mathbb{K})$ the class of IVL-algebras whose subreduct of exact elements is isomorphic to an L-algebra in $\mathbb{K}$. This is because the connection between the strong standard completeness of a core fuzzy logic L and the real-chain embedding property is only a particular case of the connection between the strong completeness w.r.t. $\mathbb{K}$ of a core fuzzy logic L and the '$\mathbb{K}$-chain embedding property'. The proof for strong completeness of IVL w.r.t. $TA(\mathbb{K})$ therefore remains completely similar to the proof for strong standard completeness of IVL.

Remark 16. In the previous remark we noted that for core fuzzy logics there is connection between the strong completeness w.r.t. a class $\mathbb{K}$ of L-chains and the '$\mathbb{K}$-chain embedding property', which was used to demonstrate the strong completeness of IVL w.r.t. $TA(\mathbb{K})$ (under the condition that L is strong complete w.r.t. $\mathbb{K}$). For core fuzzy logics L in a finite language (e.g., all axiomatic extensions of MTL), we have a similar equivalence between the finite strong completeness w.r.t. a class $\mathbb{K}$ of L-chains and the '$\mathbb{K}$-chain partial-embedding property'. Completely similarly as for strong completeness, we can use this equivalence to show the finite strong completeness of IVL w.r.t. $TA(\mathbb{K})$ (under the condition that IVL is the interval-valued companion of a core fuzzy logic L without extra connectives (and thus in a finite language) which is finite strong complete w.r.t. $\mathbb{K}$).

In particular, for a finite strong standard complete core fuzzy logic L without extra connectives, we find that its interval-valued companion IVL is finite strong standard complete. Because Ł, WCMTL, ΠMTL, BL, Π and SBL are all finite strong standard complete core fuzzy logics in a finite language (see [4,18,21,22,28]), IVŁ, IVWCMTL, IVΠMTL, IVBL, IVΠ and IVSBL are all finite strong standard complete (and therefore also standard complete). This makes that all logics in Table 2, apart from IVCPC (and IVML), are finite strong standard complete.

As witnessed in [10], it can occur that a core fuzzy logic L is complete w.r.t. a class $\mathbb{K}$ of L-chains, but not finite strong complete w.r.t. $\mathbb{K}$. In this case we do not know of a suitable characterization of completeness (in terms of a kind of embedding property). For such a core fuzzy logic L, the completeness of IVL remains an open problem.

For ML, Ł, CPC, WCMTL, ΠMTL, BL, Π and SBL it is known [4,18,21,22,28] that they are not strong standard complete. The next proposition implies that their interval-valued counterparts cannot be strong standard complete either. First we mention some notations that will be used.

[16] For core fuzzy logics with extra connectives, this remains an open problem. But not for Δ-core fuzzy logics, see Section 5.

Suppose $\mathbb{K}$ is a class of residuated lattices. Recall from Remark 15 that we defined the class $TA(\mathbb{K})$ of triangle algebras as follows: a triangle algebra $\mathcal{A}$ is an element of $TA(\mathbb{K})$ iff $\mathcal{E}(\mathcal{A})$ is isomorphic to a residuated lattice in $\mathbb{K}$. Because of Proposition 11, $TA(\mathbb{K})$ is not empty if $\mathbb{K}$ is not empty.

Furthermore, for every ML-formula ϕ, we define the IVML-formula ϕ' as follows: $\phi'(p_{i_1},\ldots,p_{i_n}) = \phi(\Box p_{i_1},\ldots,\Box p_{i_n})$, where $p_{i_1},\ldots,p_{i_n}$ are the propositional variables occurring in ϕ. For example, if ϕ is the ML-formula $((p_6 \vee p_3) \to p_{12})\,\&\,(p_3 \to \bar{0})$, then ϕ' is the IVML-formula $((\Box p_6 \vee \Box p_3) \to \Box p_{12})\,\&\,(\Box p_3 \to \bar{0})$.

Also, if χ is an ML-formula, we denote the function corresponding to χ in an expansion $\mathcal{B}$ of a residuated lattice by $f_\chi^{\mathcal{B}}$. For example, if χ is the ML-formula $(p_2 \to p_4) \wedge p_2$ (which we denote by $\chi(p_2,p_4)$) and $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0, u, 1)$ is a triangle algebra, then $f_\chi^{\mathcal{A}}$ is the binary function in A defined by $f_\chi^{\mathcal{A}}(x,y) = (x \Rightarrow y) \sqcap x$, for all x and y in A.

Proposition 17. *Suppose $\Gamma \cup \{\phi\}$ is a set of ML-formulas and $\mathbb{K}$ is a class of residuated lattices. Then $\Gamma \models_{\mathbb{K}} \phi$ iff $\Gamma' \models_{TA(\mathbb{K})} \phi'$, where $\Gamma' = \{\chi' | \chi \in \Gamma\}$.*

Proof. Suppose $\Gamma' \models_{TA(\mathbb{K})} \phi'$. Now take any residuated lattice $\mathcal{L}$ in $\mathbb{K}$ and $\mathcal{L}$-model v of Γ. We want to prove that $v(\phi) = 1$. Take any triangle algebra $\mathcal{A}$ in $TA(\mathbb{K})$ such that $\mathcal{E}(\mathcal{A})$ is isomorphic to $\mathcal{L}$. Because of Proposition 11 such a triangle algebra always exists. Let i be the mapping from $\mathcal{L}$ to $\mathcal{A}$ that maps $\mathcal{L}$ isomorphically on $\mathcal{E}(\mathcal{A})$. Then the values $i(v(p_1))$, $i(v(p_2))$, $i(v(p_3))$, ... are well-defined, and we can extend this mapping of propositional variables in $\mathcal{A}$ to an $\mathcal{A}$-evaluation v' of all IVML-formulas, in a unique way. So $v'(p_j) = i(v(p_j))$ for all propositional variables p_j. Remark now that $v'(\chi') = i(v(\chi))$ for all ML-formulas χ. Indeed, if $p_{i_1},\ldots,p_{i_n}$ are the propositional variables occurring in χ, then we find $v'(\chi'(p_{i_1},\ldots,p_{i_n})) = v'(\chi(\Box p_{i_1},\ldots,\Box p_{i_n})) = f_\chi^{\mathcal{A}}(\nu v'(p_{i_1}),\ldots,\nu v'(p_{i_n})) = f_\chi^{\mathcal{A}}(\nu i(v(p_{i_1})),\ldots,\nu i(v(p_{i_n}))) = f_\chi^{\mathcal{A}}(i(v(p_{i_1})),\ldots,i(v(p_{i_n}))) = i(f_\chi^{\mathcal{L}}(v(p_{i_1}),\ldots,v(p_{i_n}))) = i(v(\chi(p_{i_1},\ldots,p_{i_n})))$. In particular, for all ψ in Γ, we have $v'(\psi') = i(v(\psi)) = i(1) = 1$. Our assumption $\Gamma \cup \{\phi\}$ ensures that $v'(\phi') = 1$. We conclude $1 = v'(\phi') = i(v(\phi))$, which implies $v(\phi) = 1$.

Now suppose $\Gamma \models_{\mathbb{K}} \phi$, and take any triangle algebra $\mathcal{A}$ in $TA(\mathbb{K})$ and $\mathcal{A}$-model v' of Γ'. We want to prove that $v'(\phi') = 1$. Therefore we consider the $\mathcal{E}(\mathcal{A})$-evaluation v determined by $v(p_i) = v'(\Box p_i)$, for all propositional variables p_i. Then for all ML-formulas χ, we have $v(\chi) = v'(\chi')$. Indeed, if $p_{i_1},\ldots,p_{i_n}$ are the propositional variables occurring in χ, then we find $v(\chi(p_{i_1},\ldots,p_{i_n})) = f_\chi^{\mathcal{E}(\mathcal{A})}(v(p_{i_1}),\ldots,v(p_{i_n})) = f_\chi^{\mathcal{E}(\mathcal{A})}(v'(\Box p_{i_1}),\ldots,v'(\Box p_{i_n})) = f_\chi^{\mathcal{A}}(v'(\Box p_{i_1}),\ldots,v'(\Box p_{i_n})) = v'(\chi(\Box p_{i_1},\ldots,\Box p_{i_n})) = v'(\chi'(p_{i_1},\ldots,p_{i_n}))$. In particular, for all ψ in Γ, we have $v(\psi) = v'(\psi') = 1$. Our assumption ensures that $v(\phi) = 1$. We conclude $1 = v(\phi) = v'(\phi')$. $\Box$

Proposition 17 enables us to show some negative completeness results for extensions of IVML.

For example, if we choose $\mathbb{K}$ to be the class of all BL-algebras, then $TA(\mathbb{K})$ is the class of all triangle algebras $\mathcal{A}$ for which $\mathcal{E}(\mathcal{A})$ is a BL-algebra. In other words, $TA(\mathbb{K})$ is the class of all triangle algebras $\mathcal{A} = (A, \sqcap, \sqcup, *, \Rightarrow, \nu, \mu, 0_A, u, 1_A)$ satisfying $(\nu x \Rightarrow \nu y) \sqcup (\nu y \Rightarrow \nu x) = 1$ and $\nu x \sqcap \nu y = \nu x * (\nu x \Rightarrow \nu y)$ for all x and y in A. So $TA(\mathbb{K})$ is the class of all IVBL-algebras. The corresponding logic is IVBL: IVML extended with the axiom schemes $(\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi)$ and $(\Box\phi \wedge \Box\psi) \to (\Box\phi\,\&\,(\Box\phi \to \Box\psi))$. It is known that BL is not strong standard complete, so there exists a set of formulas $\Gamma \cup \{\phi\}$ such that $\Gamma \models_{\mathcal{L}} \phi$ for every standard BL-algebra $\mathcal{L}$, but not for every BL-algebra $\mathcal{L}$. Proposition 17 then allows us to deduce that $\Gamma' \models_{\mathcal{A}} \phi'$ for every pseudo-divisible standard triangle algebra $\mathcal{A}$, but not for every pseudo-divisible pseudo-prelinear triangle algebra $\mathcal{A}$. Because IVBL is sound and complete w.r.t. pseudo-divisible pseudo-prelinear triangle algebras, this means exactly that this logic is not strong standard complete.

Because ML, Ł, CPC, WCMTL, ΠMTL, Π (and every schematic extension between ΠMTL and Π) and SBL are not strong standard complete [4], we can reason in the same way as for BL and conclude that IVML, IVŁ, IVCPC, IVWCMTL, IVΠMTL, IVΠ and IVSBL are not strong standard complete either. We give an overview of the completeness results in Table 3. Between brackets are the known completeness results for the non-IV counterparts. We note that for a core fuzzy logic L that is standard com-

Table 3
Completeness of several extensions of IVML.

Logic	Standard complete	Finite strong standard complete	Strong standard complete
IVML	No (No)	No (No)	No (No)
IVMTL	Yes (Yes)	Yes (Yes)	Yes (Yes)
IVŁ	Yes (Yes)	Yes (Yes)	No (No)
IVCPC	No (No)	No (No)	No (No)
IVG	Yes (Yes)	Yes (Yes)	Yes (Yes)
IVWNM	Yes (Yes)	Yes (Yes)	Yes (Yes)
IVIMTL	Yes (Yes)	Yes (Yes)	Yes (Yes)
IVNM	Yes (Yes)	Yes (Yes)	Yes (Yes)
IVSMTL	Yes (Yes)	Yes (Yes)	Yes (Yes)
IVWCMTL	Yes (Yes)	Yes (Yes)	No (No)
IVΠMTL	Yes (Yes)	Yes (Yes)	No (No)
IVBL	Yes (Yes)	Yes (Yes)	No (No)
IVΠ	Yes (Yes)	Yes (Yes)	No (No)
IVSBL	Yes (Yes)	Yes (Yes)	No (No)

plete but not finite strong standard complete, the result for IVL would be "Unknown No No" (for standard, finite strong standard and strong standard completeness, respectively).

Now we will show a local deduction theorem for IVML and its extensions. Let **L** be an extension of IVML.

From the definition of a proof of $\Gamma \vdash_{\mathbf{L}} \phi$, we immediately obtain the following property (which is actually a property of all logical systems).

Lemma 18. *Let $\Gamma_1 \cup \Gamma_2 \cup \{\phi, \psi\}$ be a set of IVML-formulas, and **L** be an extension of IVML.*
If $\Gamma_1 \vdash_{\mathbf{L}} \phi$ and $\Gamma_2 \cup \{\phi\} \vdash_{\mathbf{L}} \psi$, then $\Gamma_1 \cup \Gamma_2 \vdash_{\mathbf{L}} \psi$.

Proof. Observe that putting the proof of $\Gamma_2 \cup \{\phi\} \vdash_{\mathbf{L}} \psi$ after the proof of $\Gamma_1 \vdash_{\mathbf{L}} \phi$, gives a proof of $\Gamma_1 \cup \Gamma_2 \vdash_{\mathbf{L}} \psi$. □

Proposition 19. *Let $\Gamma \cup \{\phi\}$ be a set of IVML-formulas, and **L** be an extension of IVML. Then $\Gamma \vdash_{\mathbf{L}} \phi$ iff $\Gamma \vdash_{\mathbf{L}} \Box\phi$.*

Proof. On the one hand, we can apply Lemma 18 with $\Gamma_2 = \emptyset$ and $\psi = \Box\phi$, because $\{\phi\} \vdash_{\mathbf{L}} \Box\phi$ (application of the □-necessitation rule).

On the other hand, we can apply Lemma 18 to $\Gamma \vdash_{\mathbf{L}} \Box\phi$ and $\{\Box\phi\} \vdash_{\mathbf{L}} \phi$ (application of the modus ponens to IVML.1). □

In a similar way we can prove the following proposition.

Proposition 20. *Let $\Gamma \cup \{\phi, \psi\}$ be a set of IVML-formulas, and **L** be an extension of IVML. Then $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$ if and only if $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} \psi$.*

Proof. In one direction, apply Lemma 18 to $\{\phi\} \vdash_{\mathbf{L}} \Box\phi$ and $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} \psi$. In the other direction, apply the lemma to $\{\Box\phi\} \vdash_{\mathbf{L}} \phi$ and $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$. □

Now we prove a so-called local deduction theorem for IVML (and its extensions), which gives a connection between $\vdash_{\mathbf{L}}$ and $\to$.

Proposition 21. *Let $\Gamma \cup \{\phi, \psi\}$ be a set of IVML-formulas, and **L** be an extension of IVML.*
Then the following are equivalent:

- $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} \psi$,
- *There is an integer n such that $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^n \to \psi$.*

Proof. Suppose $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^n \to \psi$, which is equivalent with $\Gamma \vdash_{\mathbf{L}} \Box\phi \to ((\Box\phi)^{n-1} \to \psi)$ because of ML.11. Then by an application of modus ponens we obtain $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} (\Box\phi)^{n-1} \to \psi$. Proceeding like this, we get $\Gamma \cup \{\Box\phi\} = \Gamma \cup \{\Box\phi\} \cup \{\Box\phi\} \vdash_{\mathbf{L}} (\Box\phi)^{n-2} \to \psi, \ldots$ and finally $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} \Box\phi \to \psi$ and $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} \psi$.

Now suppose $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} \psi$. This means that there is a proof of ψ, in which every line is an axiom, an element of $\Gamma \cup \{\Box\phi\}$, or an application of modus ponens, □-necessitation or monotonicity of ◇ to previous lines in the proof. We will show by induction that for all the formulas γ in the proof, there exists an integer n such that $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^n \to \gamma$. This will imply $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^n \to \psi$ for some integer n, as ψ is the last line of the proof. Remark that we can use soundness and completeness of IVML w.r.t. triangle algebras. So we know that $\vdash_{\mathbf{L}} \phi$ if ϕ holds in every triangle algebra.

We have to consider the following possibilities:

- γ is an axiom or an element of Γ. Then we have $\Gamma \vdash_{\mathbf{L}} \gamma$, which is equivalent with $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^0 \to \gamma$.
- γ is $\Box\phi$. In this case, we have $\Gamma \vdash_{\mathbf{L}} (\Box\phi) \to \gamma$.
- γ is the result of an application of modus ponens. So there are two formulas α and $\alpha \to \gamma$ earlier in the proof. By induction hypothesis, we know that there are integers k and l such that $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^k \to \alpha$ and $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^l \to (\alpha \to \gamma)$. Combining these, we find $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^{k+l} \to (\alpha \,\&\, (\alpha \to \gamma))$. As we also have $\vdash_{\mathbf{L}} (\alpha \,\&\, (\alpha \to \gamma)) \to \gamma$, we obtain $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^{k+l} \to \gamma$.
- γ is the result of an application of □-necessitation. This means γ is of the form $\Box\alpha$, where α is a formula occuring earlier in the proof. So by induction hypothesis, there is an integer k such that $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^k \to \alpha$. Applying □-necessitation, IVML.7 and modus ponens, we get $\Gamma \vdash_{\mathbf{L}} \Box((\Box\phi)^k) \to \Box\alpha$. This is equivalent with $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^k \to \Box\alpha$.
- γ is the result of an application of monotonicity of ◇. This means γ is of the form $\Diamond\alpha \to \Diamond\beta$, with $\alpha \to \beta$ a formula earlier in the proof. The induction hypothesis assures that there is an integer k such that $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^k \to (\alpha \to \beta)$. Then similarly as for □-necessitation, we find $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^k \to \Box(\alpha \to \beta)$. Because by Theorem 10 we also know $\vdash_{\mathbf{L}} \Box(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)$, $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^k \to (\Diamond\alpha \to \Diamond\beta)$. □

Summarizing the previous propositions, we see that all of the following statements are equivalent.

- There is an integer n such that $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^n \to \psi$,
- $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$,
- $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \Box\psi$,
- there is an integer n such that $\Gamma \vdash_{\mathbf{L}} (\Box\phi)^n \to \Box\psi$,
- $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} \Box\psi$,
- $\Gamma \cup \{\Box\phi\} \vdash_{\mathbf{L}} \psi$.

Remark that in IVG $\Box\phi$ and $(\Box\phi)^n$ $(n \geqslant 1)$ are provably equivalent, so for IVG and its extensions we have a stronger deduction theorem: $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$ iff $\Gamma \vdash_{\mathbf{L}} \Box\phi \to \psi$.

5. The expansion of IVML and its axiomatic extensions with Baaz's Delta

In this section we introduce IVML_Δ and show that the deduction theorem holds for this logic and its extensions. For IVMTL_Δ and its extensions, we argue that similar completeness results hold as in Section 4.

We start by proving that $(\Delta 4)$ is superfluous.

Proposition 22. *Let $(L, \sqcap, \sqcup, *, \Rightarrow, \Delta, 0, 1)$ be an ML_Δ-algebra.*[17] *Then $\Delta\Delta x = \Delta x$ and $\Delta(x * y) = \Delta x * \Delta y = \Delta x \sqcap \Delta y = \Delta(x \sqcap y)$, for all x and y in L.*

Proof. On one hand, we have $\Delta\Delta x \leqslant \Delta x$. On the other hand, we have $1 = \Delta 1 = \Delta(\Delta x \sqcup \neg\Delta x) \leqslant \Delta\Delta x \sqcup \Delta\neg\Delta x \leqslant \Delta\Delta x \sqcup \neg\Delta x$, and therefore $\Delta x = \Delta x * 1 = \Delta x * (\Delta\Delta x \sqcup \neg\Delta x) = \Delta x * \Delta\Delta x \sqcup \Delta x * \neg\Delta x = \Delta x * \Delta\Delta x \sqcup 0 \leqslant \Delta\Delta x$. To prove $\Delta(x * y) = \Delta x * \Delta y = \Delta x \sqcap \Delta y = \Delta(x \sqcap y)$, we first note that it is already known (see e.g. [18]) that $\Delta x * \Delta x = \Delta x$ and $\Delta x * \Delta y = \Delta(x * y)$ are valid for all x and y in L. Using these properties, we find $\Delta(x \sqcap y) = \Delta(x \sqcap y) * \Delta(x \sqcap y) \leqslant \Delta x * \Delta y \leqslant \Delta x \sqcap \Delta y = (\Delta x \sqcap \Delta y) * (\Delta y \sqcup \neg\Delta y) = ((\Delta x \sqcap \Delta y) * \Delta y) \sqcup ((\Delta x \sqcap \Delta y) * \neg\Delta y) \leqslant \Delta x * \Delta y \sqcup 0 = \Delta(x * y) \leqslant \Delta(x \sqcap y)$. $\Box$

Because the implicative logic (in the sense of Rasiowa [29], which can be verified easily) ML_Δ is sound w.r.t. the variety of ML_Δ-algebras, it is also strong complete w.r.t. it [13]. Therefore Proposition 22 implies that $\vdash_{ML_\Delta} \Delta\phi \to \Delta\Delta\phi$.

Definition 23. Let L be an axiomatic expansion of ML_Δ. Then we define its interval-valued companion IVL as the logic with the following axioms and deduction rules: the union of the axioms of IVML and the axioms of ML_Δ and the union of the deduction rules of IVML and the deduction rules of ML_Δ (in other words, MP, G, M$\Diamond$ and N), plus the 'box translations'[18] of all extra[19] axioms of L, plus two axioms $f(\Box\phi_1, \ldots, \Box\phi_n) \to \Box f(\Box\phi_1, \ldots, \Box\phi_n)$ and $\Delta((\phi_1 \leftrightarrow \psi_1) \,\&\, \cdots \,\&\, (\phi_n \leftrightarrow \psi_n)) \to (f(\phi_1, \ldots, \phi_n) \leftrightarrow f(\psi_1, \ldots, \psi_n))$ for every extra n-ary connective f in L.

L-algebras and IVL-algebras are defined in the usual way.

In particular, an IVML_Δ-algebra is an algebra $(A, \sqcap, \sqcup, *, \Rightarrow, v, \mu, \Delta, 0, u, 1)$ in which $(A, \sqcap, \sqcup, *, \Rightarrow, v, \mu, 0, u, 1)$ is a triangle algebra and the unary operator Δ satisfies $\Delta 1 = 1$, $\Delta x \sqcup \neg\Delta x = 1$, $\Delta(x \sqcup y) \leqslant \Delta x \sqcup \Delta y$, $\Delta x \leqslant x$ and $\Delta(x \Rightarrow y) \leqslant \Delta x \Rightarrow \Delta y$, for all x and y in L.

Note that IVMTL_Δ is IVML_Δ + pseudoprelinearity, IVBL_Δ is IVMTL_Δ + pseudodivisibility, ... (similarly as for IVBL, IVMTL, IVML, ...), and IVMTL_Δ-algebra are pseudo-prelinear IVML_Δ-algebras, IVBL_Δ-algebras are pseudo-divisible pseudo-prelinear IVML_Δ-algebras, ...

As a slightly more complex example, consider the axiomatic expansion L_1 of ML_Δ with a new connective $\sim$ and the axioms $(\phi \to \psi) \vee (\psi \to \phi)$, $\sim\sim\phi \to \phi$ and $\Delta(\phi \to \psi) \to \Delta(\sim\psi \to \sim\phi)$. Then IVL_1 is determined by the axioms and deduction rules of IVML_Δ, plus $(\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi)$, $\sim\sim\Box\phi \to \Box\phi$ and $\Delta(\Box\phi \to \Box\psi) \to \Delta(\sim\Box\psi \to \sim\Box\phi)$, plus $\sim\Box\phi \to \Box\sim\Box\phi$ and $\Delta(\phi \leftrightarrow \psi) \to (\sim\phi \leftrightarrow \sim\psi)$ (for the new connective $\sim$).

An L_1-algebra is an algebra $(L, \sqcap, \sqcup, *, \Rightarrow, \Delta, \sim, 0, 1)$ in which $(L, \sqcap, \sqcup, *, \Rightarrow, \Delta, 0, 1)$ is an ML_Δ-algebra and such that $(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1$, $\sim\sim x \Rightarrow x = 1$ and $\Delta(x \Rightarrow y) \Rightarrow \Delta(\sim y \Rightarrow \sim x) = 1$ hold for all x and y in L.

An IVL_1-algebra is an algebra $(A, \sqcap, \sqcup, *, \Rightarrow, v, \mu, \Delta, \sim, 0, u, 1)$ in which $(A, \sqcap, \sqcup, *, \Rightarrow, v, \mu, \Delta, 0, u, 1)$ is an IVML_Δ-algebra and such that $(vx \Rightarrow vy) \sqcup (vy \Rightarrow vx) = 1$, $\sim\sim vx \Rightarrow vx = 1$, $\Delta(vx \Rightarrow vy) \Rightarrow \Delta(\sim vy \Rightarrow \sim vx) = 1$, $\sim vx \Rightarrow v\sim vx = 1$ and $\Delta(x \Longleftrightarrow y) \Rightarrow (\sim x \Longleftrightarrow \sim y) = 1$ hold for all x and y in A.

Similarly as for IVML (see [32]) we can show that interval-valued companions of axiomatic expansions of ML_Δ are implicative logics and conclude that such a logic is sound and strong complete w.r.t. the variety of the corresponding algebras. The part of the proof not yet considered in [32] is $\Gamma \vdash \Delta\phi \leftrightarrow \Delta\psi$ if $\Gamma \vdash \phi \leftrightarrow \psi$ (which is proven exactly as for $\Box$) and, for every

[17] We mean that $(L, \sqcap, \sqcup, *, \Rightarrow, 0, 1)$ is a residuated lattice and that Δ satisfies $\Delta 1 = 1$, $\Delta x \sqcup \neg\Delta x = 1$, $\Delta(x \sqcup y) \leqslant \Delta x \sqcup \Delta y$, $\Delta x \leqslant x$ and $\Delta(x \Rightarrow y) \leqslant \Delta x \Rightarrow \Delta y$, for all x and y in L.

[18] Similarly as in Table 2. For example, the box translation of prelinearity is pseudo-prelinearity, the box translation of divisibility is pseudo-divisibility, and so on.

[19] With extra axioms of L, we mean the axioms of L that are different from those in ML_Δ.

extra (n-ary) connective f, $\Gamma \vdash f(\phi_1,\ldots,\phi_n) \leftrightarrow f(\psi_1,\ldots,\psi_n)$ if $\Gamma \vdash \phi_1 \leftrightarrow \psi_1, \ldots$ and $\Gamma \vdash \phi_n \leftrightarrow \psi_n$ (which is proven using Δ-necessitation, the axiom $\Delta((\phi_1 \leftrightarrow \psi_1) \,\&\, \cdots \,\&\, (\phi_n \leftrightarrow \psi_n)) \to (f(\phi_1,\ldots,\phi_n) \leftrightarrow f(\psi_1,\ldots,\psi_n))$, and modus ponens).

Proposition 24. *Let $(A,\sqcap,\sqcup,*,\Rightarrow,\nu,\mu,\Delta,0,u,1)$ be an IVML$_\Delta$-algebra. Then $\Delta\Delta x = \Delta x$, $\Delta(x \sqcup y) = \Delta x \sqcup \Delta y$, $\Delta(x * y) = \Delta x * \Delta y = \Delta x \sqcap \Delta y = \Delta(x \sqcap y)$, $\Delta x * \Delta x = \Delta x$, $\Delta x \leqslant \nu x$, $\nu\Delta x = \Delta x = \Delta\nu x$ and $\Delta(x \Longleftrightarrow y) = \Delta(\nu x \Longleftrightarrow \nu y) * \Delta(\mu x \Longleftrightarrow \mu y)$ for all x and y in A.*

Proof. The first four properties hold in each ML$_\Delta$-algebra and thus a fortiori also in each IVML$_\Delta$-algebra. Now we prove that $\Delta x \leqslant \nu\Delta x$. First note that $1 = \nu 1 = \nu(\Delta x \sqcup \neg\Delta x) = \nu\Delta x \sqcup \nu\neg\Delta x \leqslant \nu\Delta x \sqcup \neg\Delta x$. Therefore $\Delta x = \Delta x * 1 = \Delta x * (\nu\Delta x \sqcup \neg\Delta x) = \Delta x * \nu\Delta x \sqcup \Delta x * \neg\Delta x = \Delta x * \nu\Delta x \sqcup 0 \leqslant \nu\Delta x$. As the converse inequality holds as well, $\Delta x = \nu\Delta x$. We also find $\Delta x = \nu\Delta x \leqslant \nu x$, and $\Delta x = \Delta\Delta x \leqslant \Delta\nu x$ (which implies $\Delta x = \Delta\nu x$ because $\Delta\nu x \leqslant \Delta x$).

Furthermore $\Delta(x \Longleftrightarrow y) = \Delta((x \Rightarrow y) \sqcap (y \Rightarrow x)) = \Delta(x \Rightarrow y) * \Delta(y \Rightarrow x) = \Delta\nu(x \Rightarrow y) * \Delta\nu(y \Rightarrow x) = \Delta((\nu x \Rightarrow \nu y) \sqcap (\mu x \Rightarrow \mu y)) * \Delta((\nu y \Rightarrow \nu x) \sqcap (\mu y \Rightarrow \mu x)) = \Delta(\nu x \Rightarrow \nu y) * \Delta(\mu x \Rightarrow \mu y) * \Delta(\nu y \Rightarrow \nu x) \,\&\, \Delta(\mu y \Rightarrow \mu x) = \Delta(\nu x \Longleftrightarrow \nu y) * \Delta(\mu x \Longleftrightarrow \mu y)$. $\square$

As a corollary, the image of an element x under Δ is always an exact element. In particular, the subset of exact elements of an IVML$_\Delta$-algebra is closed under Δ. For each IVML$_\Delta$-algebra $\mathcal{A} = (A,\sqcap,\sqcup,*,\Rightarrow,\nu,\mu,\Delta,0,u,1)$, the subreduct $(E(\mathcal{A}),\sqcap,\sqcup,*,\Rightarrow,\Delta,0,1)$ is an ML$_\Delta$-algebra. Moreover, because $\Delta x = \Delta\nu x$, the action of Δ on the IVML$_\Delta$-algebra is determined by its action on the subset of exact elements.

As another corollary, in the definition of a pseudo-linear IVL-algebra $\mathcal{A}$ (with L an axiomatic expansion of ML$_\Delta$), the conditions $\Delta((x_1 \Longleftrightarrow y_1) * \cdots * (x_n \Longleftrightarrow y_n)) \Rightarrow (f(x_1,\ldots,x_n) \Longleftrightarrow f(y_1,\ldots,y_n)) = 1$ (for every extra n-ary connective f in $\mathcal{A}$) are automatically fulfilled (if all other conditions do hold, of course). Indeed, if $x_1 = y_1, \ldots$ and $x_n = y_n$ then $f(x_1,\ldots,x_n) \Longleftrightarrow f(y_1,\ldots,y_n) = 1$, thus $\Delta((x_1 \Longleftrightarrow y_1) * \cdots * (x_n \Longleftrightarrow y_n)) \Rightarrow (f(x_1,\ldots,x_n) \Longleftrightarrow f(y_1,\ldots,y_n)) = 1$. If $x_i \neq y_i$ for some i in $\{1,\ldots,n\}$, then $\nu x_i \neq \nu y_i$ or $\mu x_i \neq \mu y_i$ and thus $\nu x_i \Longleftrightarrow \nu y_i \neq 1$ or $\mu x_i \Longleftrightarrow \mu y_i \neq 1$. Because $\nu x_i \Longleftrightarrow \nu y_i$ and $\mu x_i \Longleftrightarrow \mu y_i$ are exact elements (which are linearly ordered by assumption), we find[20] $\Delta(\nu x_i \Longleftrightarrow \nu y_i) = 0$ or $\Delta(\mu x_i \Longleftrightarrow \mu y_i) = 0$ and therefore $\Delta(x_i \Longleftrightarrow y_i) = \Delta(\nu x_i \Longleftrightarrow \nu y_i) * \Delta(\mu x_i \Longleftrightarrow \mu y_i) = 0$. Thus also in this case, $\Delta((x_1 \Longleftrightarrow y_1) * \cdots * (x_n \Longleftrightarrow y_n)) \Rightarrow (f(x_1,\ldots,x_n) \Longleftrightarrow f(y_1,\ldots,y_n)) = 1$.

Now we can prove the deduction theorem for IVML$_\Delta$ and its axiomatic expansions.

Proposition 25. *Let **L** be an axiomatic expansion of IVML$_\Delta$ and $\Gamma \cup \{\phi,\psi\}$ a set of formulas in the language of **L**. Then the following are equivalent:*

- $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$,
- $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \psi$.

Proof. Suppose $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \psi$. Because $\{\phi\} \vdash_{\mathbf{L}} \Delta\phi$, by an application of modus ponens we obtain $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$.

Now suppose $\Gamma \cup \{\phi\} \vdash_{\mathbf{L}} \psi$. This means that there is a proof of ψ, in which every line is an axiom, an element of $\Gamma \cup \{\phi\}$, or an application of modus ponens, $\Box$-necessitation, monotonicity of $\Diamond$ or Δ-necessitation to previous lines in the proof. We will show by induction that for all the formulas γ in the proof, $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \gamma$. This will imply $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \psi$, as ψ is the last line of the proof. Remark that we can use soundness and completeness of IVML$_\Delta$ w.r.t. IVML$_\Delta$-algebras. So we know that $\vdash_{\mathbf{L}}\phi$ if ϕ holds in every IVML$_\Delta$-algebra.

We have to consider the following possibilities:

- γ is an axiom or an element of Γ. Then we have $\Gamma \vdash_{\mathbf{L}} \gamma$, which implies $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \gamma$.
- γ is ϕ. In this case, we have $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \gamma$.
- γ is the result of an application of modus ponens. So there are two formulas α and $\alpha \to \gamma$ earlier in the proof. By induction hypothesis, we know that $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \alpha$ and $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to (\alpha \to \gamma)$. Combining these, we find $\Gamma \vdash_{\mathbf{L}} (\Delta\phi \,\&\, \Delta\phi) \to (\alpha \,\&\, (\alpha \to \gamma))$. As we also have $\vdash_{\mathbf{L}} (\alpha \,\&\, (\alpha \to \gamma)) \to \gamma$ and $\Delta\phi \,\&\, \Delta\phi$ is equivalent with $\Delta\phi$, we obtain $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \gamma$.
- γ is the result of an application of $\Box$-necessitation. This means γ is of the form $\Box\alpha$, where α is a formula occuring earlier in the proof. So by induction hypothesis, $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \alpha$. Applying $\Box$-necessitation, IVML.7 and modus ponens, we get $\Gamma \vdash_{\mathbf{L}} \Box\Delta\phi \to \Box\alpha$. This is equivalent with $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \Box\alpha$ (because $\Box\Delta\phi$ is equivalent with $\Delta\phi$).
- γ is the result of an application of monotonicity of $\Diamond$. This means γ is of the form $\Diamond\alpha \to \Diamond\beta$, with $\alpha \to \beta$ a formula earlier in the proof. The induction hypothesis assures that $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to (\alpha \to \beta)$. Then similarly as for $\Box$-necessitation, we find $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \Box(\alpha \to \beta)$. Because by Theorem 10 we also know $\vdash_{\mathbf{L}} \Box(\alpha \to \beta) \to (\Diamond\alpha \to \Diamond\beta)$, $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to (\Diamond\alpha \to \Diamond\beta)$.
- γ is the result of an application of Δ-necessitation. This means γ is of the form $\Delta\alpha$, where α is a formula occuring earlier in the proof. So by induction hypothesis, $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \alpha$. Applying Δ-necessitation, $(\Delta 5)$ and modus ponens, we get $\Gamma \vdash_{\mathbf{L}} \Delta\Delta\phi \to \Delta\alpha$. This is equivalent with $\Gamma \vdash_{\mathbf{L}} \Delta\phi \to \Delta\alpha$ (because $\Delta\Delta\phi$ is equivalent with $\Delta\phi$). $\square$

[20] Here we use $\Delta x = 0$ if $x < 1$, which holds in linear ML$_\Delta$-algebras.

Let L be an axiomatic expansion of IVMTL$_\Delta$ (for example, an interval-valued companion of a Δ-core fuzzy logic). Similarly as in [34], we can use filters to show that every L-algebra is isomorphic to a subalgebra of the direct product of a system of pseudo-linear L-algebras. The idea behind the approach is the same as in [34], but there are some practical differences (comparable to the differences between BL and BL$_\Delta$ in [18]), which we mention here.

- A filter of an L-algebra is a non-empty subset F that is upward closed, and closed under $*$ and Δ. To show that the corresponding relation $\sim_F$ is a congruence on an L-algebra, the properties (one for each new connective) $\Delta((x_1 \Longleftrightarrow y_1) * \cdots * (x_n \Longleftrightarrow y_n)) \Rightarrow (f(x_1,\ldots,x_n) \Longleftrightarrow f(y_1,\ldots,y_n)) = 1$ are needed.
- The smallest filter of an L-algebra $\mathcal{A}$ containing a given filter F and a given element z is $\{v \in A | (\exists w \in F)(w * \Delta z \leqslant v)\}$. The proof is straightforward and similar to the proof in [34] and the proof of Theorem 2.4.12 in [18].
- The proof that $w_1 * w_2 \leqslant a$ if $w_1 * \Delta(vx \Rightarrow vy) \leqslant a$ and $w_2 * \Delta(vy \Rightarrow vx) \leqslant a$, is as follows: $w_1 * w_2 = w_1 * w_2 * \Delta((vx \Rightarrow vy) \sqcup (vy \Rightarrow vx)) = w_1 * w_2 * (\Delta(vx \Rightarrow vy) \sqcup \Delta(vy \Rightarrow vx)) = w_1 * w_2 * \Delta(vx \Rightarrow vy) \sqcup w_1 * w_2 * \Delta(vy \Rightarrow vx) \leqslant w_1 * \Delta(vx \Rightarrow vy) \sqcup w_2 * \Delta(vy \Rightarrow vx) \leqslant a \sqcup a = a$.

This decomposition theorem for IVMTL$_\Delta$-algebras allows us to strengthen the (general) strong completeness of IVMTL$_\Delta$ (and its axiomatic extensions) to pseudo-chain strong completeness.

For several interval-valued companions of Δ-core fuzzy logics, we can prove strong standard completeness in an analogous way as explained in Theorem 14 and Remark 15.

Theorem 26. *Let L be a Δ-core fuzzy logic (with[21] k extra connectives $f_1,\ldots,f_k$) that is strong complete w.r.t. a class $\mathbb{K}$ of L-chains. Then its interval-valued companion IVL is strong complete w.r.t. $TA(\mathbb{K})$ (as defined in Remark 15).*

Note that the case of strong standard completeness is obtained by choosing $\mathbb{K}$ as the class of standard L-chains.

Proof. Suppose $\Gamma \cup \{\phi\}$ is a set of formulas in the language of IVL, and $\Gamma \nvdash_{IVL} \phi$. We need to prove that there exists an IVL-algebra $\mathcal{C}$ in $TA(\mathbb{K})$ and a $\mathcal{C}$-model e of Γ such that $e(\phi) < 1_{\mathcal{C}}$.

By the strong pseudo-chain completeness of IVL, we already know there exists a pseudo-linear IVL-algebra $\mathcal{A}$ in $TA(\mathbb{K})$ and an $\mathcal{A}$-model e of Γ such that $e(\phi) < 1_{\mathcal{A}}$. Similarly as in Theorem 14, we can assume it is at most countably generated. The subreduct $\mathcal{E}(\mathcal{A})$ consisting of the exact elements of $\mathcal{A}$ is a linear L-algebra (here, the properties $f(\Box\phi_1,\ldots,\Box\phi_n) \to \Box f(\Box\phi_1,\ldots,\Box\phi_n)$ are used). Because L is a Δ-core fuzzy logic that is strong complete w.r.t. a class $\mathbb{K}$, a countably generated linear L-algebra is embeddable in an L-chain from $\mathbb{K}$. Let i denote an embedding from the reduct $\mathcal{E}(\mathcal{A}) = (D, \sqcap_D, \sqcup_D, *_D, \Rightarrow_D, \Delta_D, f_{1D}, \ldots, f_{kD}, 0_A, 1_A)$, in which $D = E(\mathcal{A})$, in the L-chain $\mathcal{B} = (B, \min, \max, \circ, \Rightarrow_\circ, \Delta_B, f'_1, \ldots, f'_k, 0, 1)$ from $\mathbb{K}$.

Now we define an IVL-algebra $\mathcal{C}$ in $TA(\mathbb{K})$ and a mapping j from $\mathcal{A}$ to $\mathcal{C}$ in the following way: $\mathcal{C} := (C, \inf, \sup, \odot, \rightsquigarrow, p_v, p_h, \Delta_C, f_{1C}, \ldots, f_{kC}, [0,0], [0,1], [1,1])$, with

- $C = Int(\mathcal{B})$
- $\inf([x_1,x_2],[y_1,y_2]) = [\min(x_1,y_1), \min(x_2,y_2)]$,
- $\sup([x_1,x_2],[y_1,y_2]) = [\max(x_1,y_1), \max(x_2,y_2)]$,
- $[x_1,x_2] \odot [y_1,y_2] = [x_1 \circ y_1, \max(x_1 \circ y_2, x_2 \circ y_1, x_2 \circ y_2 \circ i(\mu(u * u)))]$,
- $[x_1,x_2] \rightsquigarrow [y_1,y_2] = [\min(x_1 \Rightarrow_\circ y_1, x_2 \Rightarrow_\circ y_2), \min(x_1 \Rightarrow_\circ y_2, (x_2 \circ i(\mu(u * u))) \Rightarrow_\circ y_2)]$,
- $p_v([x_1,x_2]) = [x_1,x_1]$,
- $p_h([x_1,x_2]) = [x_2,x_2]$,
- $\Delta_C([x_1,x_2]) = [\Delta_B x_1, \Delta_B x_1]$,
- $j(x) = [i(vx), i(\mu x)]$, and
- $f_{lC}(j(x_1),\ldots,j(x_{n_l})) = j(f_l(x_1,\ldots,x_{n_l}))$ for elements in the image of j (for other elements in C, there are two possibilities: $f_{lC}([x_1,x_1],\ldots,[x_{n_l},x_{n_l}]) = [f'_l(x_1,\ldots,x_{n_l}), f'_l(x_1,\ldots,x_{n_l})]$; for l-tuples not of this form, the value can be chosen freely).

Similarly as in Theorem 14, we can prove that j is injective and that it is an homomorphism for inf, sup, $\odot$, $\rightsquigarrow$, p_v and p_h. Now we show that it is also a homomorphism for Δ and the extra connectives. Indeed, $j(\Delta x) = [i(v\Delta x), i(\mu\Delta x)] = [i(\Delta v\varpi(\Delta x)] = [i(v\Delta x), i(\mu\Delta x)] = [i(\Delta vx), i(\Delta vx)] = [\Delta_B i(vx), \Delta_B i(vx)] = \Delta_C([i(vx), i(\mu x)]) = \Delta_C(j(x))$. For the extra connectives, j is a homomorphism by definition. Remark that $\mathcal{C}$ is an IVL-algebra even though the image of some elements was chosen freely. This is because in a pseudo-linear IVL-algebra, the conditions on the extra connectives involve only elements on the diagonal.

Now remark that e', defined by $e'(\psi) = j(e(\psi))$, is a $\mathcal{C}$-model of Γ such that $e'(\phi) < 1$, which concludes the proof. $\square$

With a completely similar proof, we can also show the following theorem.

Theorem 27. *Let L be a Δ-core fuzzy logic (with[22] k extra connectives $f_1,\ldots,f_k$) that is finite strong complete w.r.t. a class $\mathbb{K}$ of L-chains. Then its interval-valued companion IVL is finite strong complete w.r.t. $TA(\mathbb{K})$ (as defined in Remark 15).*

[21] The number of extra connectives can also be infinite (but countable). The proof is exactly the same.

[22] The number of extra connectives cannot be infinite in this case, because the language has to be finite.

We can prove Proposition 17 also in the case for axiomatic expansions of ML_Δ.

Proposition 28. *Let* **L** *be an axiomatic expansion of* ML_Δ, $\Gamma \cup \{\phi\}$ *a set of formulas in the language of* **L** *and* $\mathbb{K}$ *is a class of* **L**-*algebras. Then* $\Gamma \models_{\mathbb{K}} \phi$ *iff* $\Gamma' \models_{TA(\mathbb{K})} \phi'$, *where* $\Gamma' = \{\chi' | \chi \in \Gamma\}$ *(where* ϕ' *is defined as before Proposition 17).*

From Theorem 26 and Proposition 28, and using the fact that a core fuzzy logic is (finite) strong complete iff its Δ-expansion is (finite) strong complete [4], we can conclude that Table 3 can be copied for the Δ-expansions of the included logics.

6. Conclusion and future work

In this paper, we have shown that the strong and finite strong standard completeness of MTL can be transferred successfully to their interval-valued counterparts. More generally, if an axiomatic extension of MTL is (finite) strong standard complete, then its interval-valued counterpart is also (finite) strong standard complete. Just like the classical standard completeness theorems stress the importance of fuzzy logics on the unit interval, our results reveal that the triangularization of the unit interval plays a similar role for interval-valued fuzzy logics, and can be endowed with analogous properties.

We also gave a local deduction theorem for IVML and its extensions.

In Section 5 we proved similar completeness results and a deduction theorem for interval-valued fuzzy logics expanded with Baaz's Delta.

An open problem for future work is to prove or disprove the standard completeness of the interval-valued counterparts of core fuzzy logics that are standard complete but not finite strong standard complete. A possible approach may be to use general methods like those in [27] and try to adapt them such that they can be used for interval-valued logics.

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