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Estimating incomplete information in group decision making: A framework of granular computing

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A B S T R A C T

A general assumption in group decision making scenarios is that of all individuals possess accurate knowledge of the entire problem under study, including the abilities to make a distinction of the degree up to which an alternative is better than other one. However, in many real world scenarios, this may be unrealistic, particularly those involving numerous individuals and options to choose from conflicting and dynamics information sources. To manage such a situation, estimation methods of incomplete information, which use own assessments provided by the individuals and consistency criteria to avoid discrepancy, have been widely employed under fuzzy preference relations. In this study, we introduce the information granularity concept to estimate missing values supporting the objective of obtaining complete fuzzy preference relations with higher consistency levels. We use the concept of granular preference relations to form each missing value as a granule of information in place of a crisp number. This offers the flexibility that is required to estimate the missing information so that the consistency levels related to the complete fuzzy preference relations are as higher as possible.

1. Introduction

Group decision making is characterized as a situation when individuals, from a set of possible options, make a choice collectively [1–5], which is here no longer attributable to any single individual but the whole group because all of them contribute to the outcome. In such a situation, most of the existing approaches have traditionally supposed that all the individuals have the necessary knowledge of the problem at hand to make a distinction of the preference degree up to which an option is more suitable than other [6,7]. However, there exist many problems where this assumption may be idealistic. In [8], it was proved that "increasing the intensity of conflict in a multicriteria comparison increases the likelihood that decision makers consider two alternatives as incomparable", resulting in incomplete information. In particular, in group decision making problems implicating a considerable amount of individuals and options to select from dynamic and contradictory information sources [9], as, for instance, the social network environments [10–13], it is very common that some of the individuals, even all of them, do not offer all the information required. Therefore, it has been necessary the development of approaches addressing the existence of incomplete information [14, 15].

Given the fact that the attempt to complete assessments between pair of options is easier than providing membership degrees to all the options in an only one step (it means the individuals can evaluate each option in contrast to all the others on the whole), the most usual representation format used by the individuals to provide their assessments is that of preference relations [16]. In addition, among the existing types of preference relations [17,18], fuzzy preference relations are the most well-known given their ability to model decision processes and their usefulness and capacity to aggregate individuals' assessments into group ones [1,19]. On the other hand, a drawback of preference relations is that of they generate more information than is actually needed (the individuals must compare every option with all the other ones) and, therefore, the likelihood of obtaining incomplete information is higher than using other representation formats, namely, preference orderings or utility values [20].

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Among the existing procedures for dealing with incomplete information in preference relations, those trying to estimate missing values are the most used [15]. On the one hand, we can find methods that estimate missing values in group decision making by using the information given by the rest of individuals along with aggregation procedures [21]. The drawbacks of these approaches is that they require several individuals to estimate the missing information of a particular one, which in conjunction with notable difference between the individuals' preferences could led to the estimation of information not naturally compatible with the rest of the individual's information. On the other hand, we can find methods that estimate missing values using just the own preferences given by the individual. In particular, the methods based on consistency criteria that estimate the individuals' incomplete information using only her/his own evaluations have been satisfactorily employed in group decision making under preference relations [22] (for more details we refer the reader to [15]).

Recently, a promising, innovative, and interesting direction is to pursue building and conceptualizing models formed as granular models [23], which may be realized as generalizations of the existing numeric models. A granular model is constructed at a higher level of abstraction and in this way becomes capable of coping with the essentials of the system modeled.

The objective of this study is to present how to generalize the existing numeric methods dealing with incomplete information to their granular methods. In particular, we present a granular estimation procedure of missing information in group decision making having the procedure proposed in [22] as its numeric counterpart. To do so, we introduce a distribution (allocation) of information granularity [24], which has been already applied successfully to increase both the consensus and the consistency in this kind of problems [25], as an essential factor to complete the missing information when the individuals verbalize their opinions via fuzzy preference relations. Then, distinct from the existing approaches dealing with missing information, we assume the missing values of a fuzzy preference relation are granular instead of numeric. It means that the missing values are considered as information granules [26] as an alternative for numeric values. Therefore, we introduce in the granular preference relation a granularity level that supplies a level of flexibility that is used to complete the missing values. This granular concept is employed to optimize (maximize) an optimization criterion, which is here associated with the individual's consistency, that is, the missing values are estimated with the purpose of increasing the consistency related to the complete fuzzy preference relation.

This study is structured in a bottom-up way and made self-contained. We structure it upon the well-known ideas of group decision making problems and recall a way in which missing information of fuzzy preference relation may be estimated (see Section 2). It uses a consistency criterion to quantify the quality of the estimated missing information. In Section 3, we discuss a way in which missing information of fuzzy preference relations may be estimated through a distribution of information granularity. Strong attention is given to the usage of the component of information granularity in the estimation of the missing values. Three experiments are reported in Section 4. Conclusions and future studies are offered in Section 5.

2. Background

We recall the idea of a fuzzy preference relation and highlight its main characteristics. We center our attention on the consistency related to fuzzy preference relations and look into a way in which missing information may be estimated when they are used.

2.1. Fuzzy preference relations

In the setting of this study, group decision making is a kind of participatory process in which more than one individual, $E = \{e_1, \ldots, e_m\}$, discuss a problem collectively, consider a collection of options, $O = \{o_1, \ldots, o_n\}$, to solve the problem and evaluate them. To do so, two processes are carried out sequentially. The first one, the consensus process [27,28], is a creative and dynamic manner of achieving agreement among all individuals of the group, which are committed to finding a solution that every individual may actively support, or at least may accept. This guarantees that all concerns, ideas and opinions, are taken into account. The second one, the selection process [22], obtains the final solution in consonance with the evaluations provided. As a result, we arrive at a rank of options from best to worst to solve the problem.

A fundamental issue in that type of problems is the way in which the evaluations provided by the individuals are represented. To do so, as we have already mentioned, fuzzy preference relations have been widely employed.

Definition 1. A fuzzy preference relation P on a set of options O is a fuzzy set on the Cartesian product $O \times O$, that is, it is characterized by a membership function $\mu_P : O \times O \to [0, 1]$.

A fuzzy preference relation P is commonly described by a $n \times n$ matrix $P = (p_{ij})$. In this representation, $p_{ij} = \mu_P(x_i, x_j)$ is the degree in which the option o_i is preferred to the option o_j. In particular, $p_{ij} = 0.5$ means indifference between both options ($o_i \sim o_j$), $p_{ij} = 1$ signifies that the option o_i is entirely preferred to the option o_j, and $p_{ij} > 0.5$ signifies the option o_i is preferred to the option o_j ($o_i \succ o_j$). Furthermore, the elements of the principal diagonal, that is, p_{ii}, are usually written as '−' because they are not important here [29].

Many decision making frameworks suppose that individuals can express evaluations between any pair of options. However, it is not all the time possible and, therefore, we have to address the problem of missing information. In a fuzzy preference relation, an entry with a missing value does not mean lack of preference of one option over other one. This can be due to the incapacity of an individual to measure the preference degree of one option over other one. Therefore, if an individual cannot provide the value of p_{ij} due to she/he does not know how better the option o_i is over the option o_j, this does not signify that the agent chooses both options with equal intensity.

These situations are characterized by the concept of an incomplete fuzzy preference relation, which was defined in [22].

Definition 2. A function $f : X \to Y$ is partial when not every element in the set X necessarily maps onto an element in the set Y. However, when every element from the set X maps onto one element of the set Y, then, in this case, we have a total function.

Definition 3. A preference relation P on a set of options O with a partial membership function is an incomplete preference relation.

2.2. Consistency

Undoubtedly decision making is a complex task. It is common that individuals' evaluations do not verify properties that a fuzzy preference relation must satisfy. Consistency, which is related to the transitivity property, is one of them [22]. However, none kind of consistency property is entailed by Definition 1 and, therefore, a fuzzy preference relations could have entries taking contradictory values, which could lead to wrong decisions [22,30].

To avoid it, the fuzzy preference relations should satisfy one of the different properties that have been proposed [31]. Given the fact that, for a fuzzy preference relation, the additive transitivity is seen as the parallel concept of the consistency property introduced by Saaty for a multiplicative preference relation [31], a methodology using this property was proposed in [22] for verifying the consistency associated with a fuzzy preference relation. It is founded on the mathematical formulation of the additive transitivity [19]:

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5), \forall i, j, k \in \{1, \ldots, n\} \quad (1)$$

Additive transitivity entails additive reciprocity, that is, as $p_{ii} = 0.5\ \forall i$, we have that $p_{ij} + p_{ji} = 1$, $\forall i, j \in \{1, \ldots, n\}$, if we make $k = i$ in Eq. (1). As a consequence, we may rewrite Eq. (1) as follows:

$$p_{ik} = p_{ij} + p_{jk} - 0.5, \forall i, j, k \in \{1, \ldots, n\} \quad (2)$$

In [22], the authors used Eq. (1) to estimate the value of an entry via other entries in a fuzzy preference relation. In particular, using an intermediate option o_j, we may estimated the value of p_{ik} ($i \neq k$) in three different ways [22]:

- We estimate the following value from $p_{ik} = p_{ij} + p_{jk} - 0.5$:

$$(ep_{ik})^{j1} = p_{ij} + p_{jk} - 0.5 \quad (3)$$

- We estimate the following value from $p_{jk} = p_{ji} + p_{ik} - 0.5$:

$$(ep_{ik})^{j2} = p_{jk} - p_{ji} + 0.5 \quad (4)$$

- We estimate the following value from $p_{ij} = p_{ik} + p_{kj} - 0.5$:

$$(ep_{ik})^{j3} = p_{ij} - p_{kj} + 0.5 \quad (5)$$

We then obtain the estimated value of p_{ik} as follows:

$$ep_{ik} = \frac{\sum_{j=1; j \neq i,k}^{n} \left((ep_{ik})^{j1} + (ep_{ik})^{j2} + (ep_{ik})^{j3}\right)}{3(n-2)} \quad (6)$$

In the case that $(ep_{ik})^{jl} = p_{ik}\ \forall j, l$, the given information is completely consistent. However, individuals are not all the time fully consistent. Hence, the evaluation given by an individual may not satisfy Eq. (1). In such a case, some of the estimated values $(ep_{ik})^{jl}$ cannot pertain to the range [0, 1]. From Eqs. (3), (4) and (5), we note that the maximum value of any $(ep_{ik})^{jl}$ ($l \in \{1, 2, 3\}$) is equal to 1.5 while the minimum value is equal to −0.5. Therefore, the error between an evaluation and its estimated one in [0, 1] is calculated as follows [22]:

$$\varepsilon p_{ik} = \frac{2}{3} \cdot |ep_{ik} - p_{ik}| \quad (7)$$

The consistency degree cd_{ik} associated with the entry p_{ik} is then obtained as follows:

$$cd_{ik} = 1 - \varepsilon p_{ik} \quad (8)$$

When $\varepsilon p_{ik} = 0$, then $cd_{ik} = 1$, which means there is consistency. The higher the value of εp_{ik} is, the lower the value of cd_{ik} is, and the more inconsistent p_{ik} is concerning the remaining information.

The consistency degrees related to the fuzzy preference relation and the individual options were then defined as follows [22]:

- The consistency degree, cd_i, associated with a given option o_i is calculated as:

$$cd_i = \frac{\sum_{k=1; i \neq k}^{n} (cd_{ik} + cd_{ki})}{2(n-1)} \quad (9)$$

- The consistency degree, cd, associated with a fuzzy preference relation is calculated as:

$$cd = \frac{\sum_{i=1}^{n} cd_i}{n} \quad (10)$$

The higher the value of cd is, the more consistent a fuzzy preference relation is. In particular, when cd is equal to 1, the fuzzy preference relation is fully consistent.

2.3. Estimation procedure of incomplete information

In [22], the authors presented an iterative approach for estimating the incomplete information of a fuzzy preference relation using Eq. (3), Eq. (4) and Eq. (5). Here, we recall its two steps:

1. Missing values to be estimated in each iteration. In the step h, EMV_h denotes the subset of missing values, MV, that we may estimate (by definition, $EMV_0 = \emptyset$). Its definition is:

$$EMV_h = \left\{ (i,k) \in MV \setminus \bigcup_{l=0}^{h-1} EMV_l \mid i \neq k \wedge \exists j \in \{H_{ik}^{h1} \cup H_{ik}^{h2} \cup H_{ik}^{h3}\} \right\} \quad (11)$$

$$A = \{(i,k) \mid i, k \in \{1, \ldots, n\} \ \wedge \ i \neq k\} \quad (12)$$

$$MV = \{(i,k) \in A \mid p_{ik} \text{ is unknown}\} \quad (13)$$

$$EV = A \setminus MV \quad (14)$$

$$H_{ik}^{h1} = \left\{ j \mid (i,j), (j,k) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \quad (15)$$

$$H_{ik}^{h2} = \left\{ j \mid (j,i), (j,k) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \quad (16)$$

$$H_{ik}^{h3} = \left\{ j \mid (i,j), (k,j) \in \left\{ EV \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \quad (17)$$

being A the set of all pair of options, MV the set of pairs of options in which the preference degree of the first option over the second one is unknown or missing, EV the set of pairs of options whose preference degrees are provided by the individual, H_{ik}^{h1}, H_{ik}^{h2}, H_{ik}^{h3}, are the sets of the intermediate option $o_j (j \neq i, k)$ that can be used to estimate the preference degree $p_{ik} (i \neq k)$ in the step h using Eqs. (3), (4) and (5).
The procedure stops when $EMV_{maxIter} = \emptyset$ ($maxIter > 0$) because we may not estimate more missing values. In addition, in the case that $\bigcup_{l=0}^{maxIter} EMV_l = MV$, all missing values of the incomplete fuzzy preference relation have been estimated and, as a consequence, the procedure has successful estimated all the missing values.
2. Estimating a given missing value. In the step h, $estimate\_p(i,k)$ is applied to estimate a value p_{ik} with $(i,k) \in EMV_h$. It is estimated as the average of all the estimated values obtained according to all the possible intermediate options o_j by means of Eqs. (3), (4) and (5).

Function *estimate_p(i, k)* **is**
  $cp_{ik}^1 = 0$; $cp_{ik}^2 = 0$; $cp_{ik}^3 = 0$; $\mathcal{K} = 0$;
  $cp_{ik}^1 = \left(\left(\sum_{j \in H_{ik}^{h1}} cp_{ik}^{j1}\right) / \#H_{ik}^{h1}\right)$;
  if $H_{ik}^{h1} \neq 0$ **then**
    $\mathcal{K} = \mathcal{K} + 1$;
  end
  $cp_{ik}^2 = \left(\left(\sum_{j \in H_{ik}^{h2}} cp_{ik}^{j2}\right) / \#H_{ik}^{h2}\right)$;
  if $H_{ik}^{h2} \neq 0$ **then**
    $\mathcal{K} = \mathcal{K} + 1$;
  end
  $cp_{ik}^3 = \left(\left(\sum_{j \in H_{ik}^{h3}} cp_{ik}^{j3}\right) / \#H_{ik}^{h3}\right)$;
  if $H_{ik}^{h3} \neq 0$ **then**
    $\mathcal{K} = \mathcal{K} + 1$;
  end
  $cp_{ik} = (1/\mathcal{K}) \cdot \left(cp_{ik}^1 + cp_{ik}^2 + cp_{ik}^3\right)$;
  return cp_{ik}
end

Algorithm 1: Iterative estimation procedure

$EMV_0 = \emptyset$;
$h = 1$;
while $EMV_h \neq \emptyset$ **do**
  foreach $(i, k) \in EMV_h$ **do** `estimate_p(i,k)`;
  $h = h + 1$;
end

In summary, given a particular incomplete fuzzy preference relation, we may estimate its missing values using Algorithm 1.

Remark 1. This procedure estimates the missing values using only the preference values given by the individual. By doing this, the procedure assures that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by the individual [22]. Therefore, if the preference values provided by the individual are inconsistent, the estimated values could also be inconsistent. In such a case, the preference values given by the individual must also be modified if we want to obtain a consistent fuzzy preference relation. To do so, the *estimate_p*(i, k) function should be applied for all the preference values, and not only for the missing values.

3. Estimating missing information through an allocation of information granularity

In this section, we describe how an allocation of information granularity may help to complete a fuzzy preference relation, which has missing values, with the higher possible consistency degree. To address this quest, we introduce an idea of a granular preference relation, which is a generalization of a fuzzy preference relation that is constructed due to a distribution of information granularity [32].

Information granularity [24] is used here as a very important design asset that may be exploited as a means to complete incomplete fuzzy preference relations of higher consistency bringing into a picture the point of values that are non-numeric and quantifying their nature by means of information granules. That is, we give up on the precise numeric values forming the entries of the fuzzy preference relations and make them granular by accepting information granules and allocating a predetermined granularity level to them in order that the granular preference relations constructed in this way "cover" as many values as possible. This position gives rise to the allocation on information granularity [32], which is another essential principle of Granular Computing [33,34]. In our setting, the allocation of information granularity elevates the fuzzy preference relations to a new level called granular preference relations.

We employ the symbol $\boldsymbol{G}(P)$ to stress that we use granular preference relations, being $\boldsymbol{G}(\cdot)$ a particular formalism of information granules [35]. Note that it is a general expression and that we are not limited to any specific granular formalism used here, namely, probability density functions, fuzzy sets, or intervals, to cite some alternatives that are usually encountered.

Concerning the estimation of missing values via a granular preference relation, there are two crucial aspects to be considered: (i) how to allocate the information granularity to the entries with missing values, and (ii) how to exploit the information granularity to complete incomplete fuzzy preference relations of higher granularity. Both aspects are described in detail in what follows.

3.1. Allocation of information granularity

The information granularity may be distributed in some different ways [24]. For clarity of the presentation, we use here a uniform allocation (distribution), in which all estimated values are treated similarly and become substituted by intervals of the same length. It means that we use intervals as information granules, and, therefore, $\boldsymbol{G}(P) = \boldsymbol{I}(P)$, where $\boldsymbol{I}(\cdot)$ denotes a family of intervals. That is, we take advantage of the estimation procedure described in Section 2.3 and augment it to some extent in order that it becomes adjusted. By these actions, we completely accept that the current knowledge source should be taken with a pinch of salt and the results provided by the estimation procedure should reflect the partial relevance of the procedure in the situation at present. This effect is quantified by making the estimated values granular (namely, more general and abstract) in order that the model may be built around the conceptual framework provided up to now. In addition, we symmetrically distribute the intervals around the estimated values.

3.2. Exploiting information granularity to estimate missing values

In the granular model of fuzzy preference relations, we need to consider that the estimated values are adjusted within the limits offered by the granularity level that is admissible with the purpose of increasing the consistency related to the fuzzy preference relation. Hence, the granularity level is employed to estimate the missing values so that the complete fuzzy preference relation is of higher consistency. We bring about this improvement at the level of each individual. This effect is quantified by the following performance index:

$$Q = \frac{1}{m} \sum_{l=1}^{m} cd^l \tag{18}$$

where m is the number of individuals participating in the decision process and cd^l represents the consistency degree related to the fuzzy preference relation expressed by the individual e_l, which is calculated using Eq. (10).

This optimization problem is willing to maximize the above performance index. It reads as follows:

$$\text{Max}_{P^1, P^2, \ldots, P^m \in \boldsymbol{I}(P)} Q \tag{19}$$

This optimization task is performed for all granular preference relations that are admissible on account of the introduced granularity level. Given the fact that this task is complicated (the search

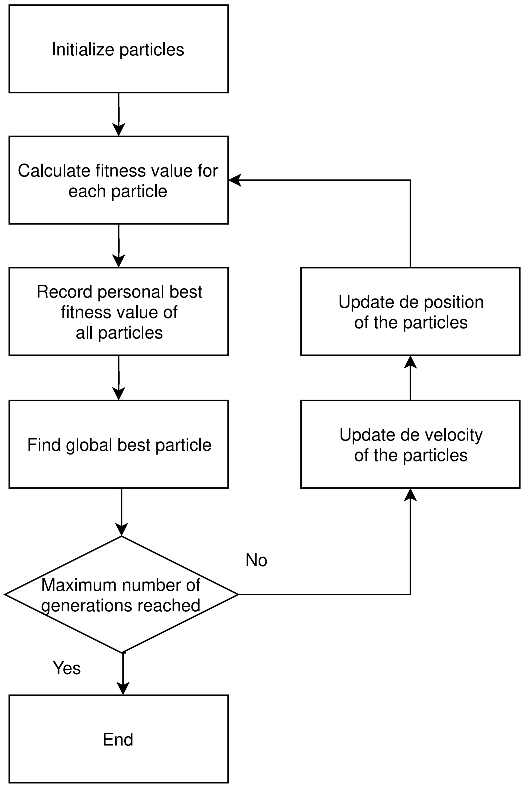


Fig. 1. Particle swarm optimization flowchart.

space is quite large as it is composed of $\boldsymbol{I}(P)$), it requires the usage of advanced global optimization techniques. In particular, this optimization task is achieved via the particle swarm optimization [36,37]. For this problem, this technique is viable since it provides a considerable level of optimization flexibility and is not accompanied by a prohibitive computational overhead level.

The particle swarm optimization is inspired by the foraging behavior of animals. It uses a swarm of particles to model the animals and to search the location of food (optimal solution) in a solution space that is n-dimensional [38] (see Fig. 1). Each particle i is composed of a velocity and a position, which are represented by $\boldsymbol{v}_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ and $\boldsymbol{x}_i = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$, respectively. In addition, each particle i has the individual memory of its best historical position $\boldsymbol{x}_i^{lbest}$ and its best fitness value y_i^{lbest}. Moreover, the best individual memory $\boldsymbol{x}^{gbest}$ is broadcast across the whole population.

In each generation t, each particle adapts its position and search pattern in the dth dimension based on its individual memory $\boldsymbol{x}_i^{lbest}$ and the global memory $\boldsymbol{x}^{gbest}$ as follows:

$$v_{id}(t+1) = \omega(t)\cdot v_{id}(t)+c_1\cdot r_{1d}\cdot(x_{id}^{lbest}-x_{id}(t))+c_2\cdot r_{2d}\cdot(x_d^{gbest}-x_{id}(t)) \tag{20}$$

$$x_{id}(t+1) = x_{id}(t) + v_{id}(t+1) \tag{21}$$

where c_1 is defined as a cognitive acceleration coefficient and c_2 is defined as a social acceleration coefficient. According to the values of c_1 and c_2, different attention is paid to the global search and the local search. r_{1d} and r_{2d} represent random numbers that are generated in [0, 1]. Finally, the local and global search ability of the particles in the generation t is balanced by the inertia weight $\omega(t)$ [39]. For local search, a small value is more suitable while a large value boosts the global search. Its value is usually decreased linearly according to [40]:

$$\omega(t) = (\omega_{start} - \omega_{end})\cdot\frac{t_{max}-t}{t_{max}} + \omega_{end} \tag{22}$$

where ω_{start} is the initial value of ω and ω_{end} is its final value, the current generation number and the maximum generation number are represented by t and t_{max}, respectively, and $\omega(t)$ is the value of ω in the current generation.

In the particle swarm optimization, a notable aspect is that of establishing an association between the problem's solution and the particle's representation. In our setting, a vector models each particle, assuming each entry of the vector a value between 0 and 1. In essence, if m individuals are part of the group, the vector is composed of $\sum_{l=1}^{m} \#MV^l$ entries, being $\#MV^l$ the number of missing values encountered in the incomplete fuzzy preference relation expressed by the individual e_l.

Let us suppose a granularity level $\alpha \in [0, 1]$, an incomplete fuzzy preference relation P expressed by an individual, and a missing entry p_{ij} of P. Then, the granularity level α implies in this entry of $\boldsymbol{I}(P)$ an interval of admissible values that is calculated as follows:

$$[I_{start}, I_{end}] = [\mathrm{Max}(0, cp_{ij} - \alpha/2), \mathrm{Min}(cp_{ij} + \alpha/2, 1)] \tag{23}$$

As an illustration example, we suppose cp_{ij} is 0.71. In addition, the corresponding component of the particle x is 0.8, and the level of granularity α is 0.4. Using Eq. (23), we get that the corresponding interval to x is equal to $[I_{start}, I_{end}] = [0.51, 0.91]$. Then, using the expression $I_{start} + (I_{end} - I_{start})\cdot x$ we obtain that the new value of cp_{ij} is equal to 0.83.

The other important aspect in this optimization technique is the definition of the fitness function, which assesses the quality of each particle during the successive generations. In our setting, we aim to maximize the consistency associated with the fuzzy preference relation. Consequently, the fitness function, f, related to the particle is:

$$f = Q \tag{24}$$

where Q is the performance index introduced in Eq. (18). The higher the value returned by the fitness function is, the better the particle is.

The steps of the proposed methodology to estimate missing information in group decision making are illustrated in Fig. 2.

4. Experimental studies

We illustrate the proposal and test its performance in this section by presenting some examples. In all of them, the particle swarm optimization was applied using these values of the parameters:

- The swarm was composed of 100 particles. Given the fact that similar outcomes were achieved in different runs of the particle swarm optimization, we found that this size produces "stable" outcomes.
- The number of generations was equal to 1000. This value was chosen because the same values reported by the fitness function were observed after this number of generations.
- c_1 and c_2 were set to 2 as this value is usually used in the existing literature [41–43].
- ω_{start} was set to 0.9 and ω_{end} was set to 0.4 as we usually encounter these values in the existing literature [40].

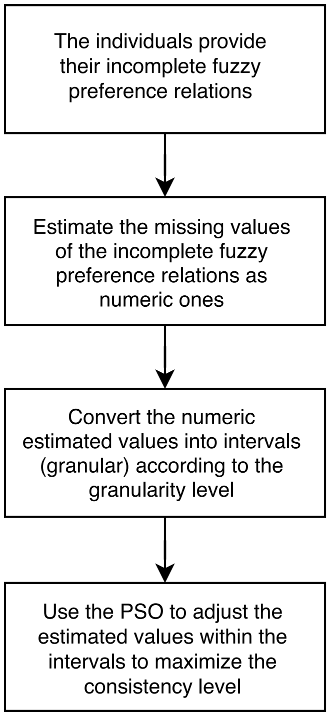


Fig. 2. Proposed methodology flowchart.

4.1. First study

In the first study, a low number of options and individuals is assumed for the sake of simplicity. Four individuals $E = \{e_1, e_2, e_3, e_4\}$ express their evaluations over a collection of five options $O = \{o_1, o_2, o_3, o_4, o_5\}$ by means of these incomplete fuzzy preference relations:

$$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ x & - & x & x & 0.20 \\ x & x & - & 0.30 & x \\ x & 0.80 & x & - & x \\ x & x & x & x & - \end{pmatrix}$$

$$P^2 = \begin{pmatrix} - & x & x & x & x \\ 0.20 & - & 0.30 & x & 0.30 \\ 0.70 & x & - & x & x \\ 0.60 & 0.10 & x & - & 0.90 \\ 0.8 & x & 1.00 & 0.30 & x \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.10 & 0.90 & x & 0.70 \\ 0.10 & - & 0.80 & x & 0.30 \\ 0.40 & x & - & x & 0.30 \\ x & 0.10 & x & - & 0.90 \\ 0.90 & x & 0.10 & x & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & x & x & x & x \\ 0.10 & - & x & 0.90 & x \\ 0.30 & x & - & x & x \\ 0.50 & x & 0.40 & - & x \\ 0.30 & 0.10 & x & x & - \end{pmatrix}$$

On the one hand, if we apply the estimation procedure presented in Section 2.3, the following complete fuzzy preference relations are obtained (the estimated values are in bold):

$$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ \mathbf{0.10} & - & \mathbf{0.35} & \mathbf{0.54} & 0.20 \\ \mathbf{0.22} & \mathbf{0.34} & - & 0.30 & \mathbf{0.35} \\ \mathbf{0.58} & 0.80 & \mathbf{0.59} & - & \mathbf{0.47} \\ \mathbf{0.43} & \mathbf{0.46} & \mathbf{0.49} & \mathbf{0.43} & - \end{pmatrix}$$

$$P^2 = \begin{pmatrix} - & \mathbf{0.00} & \mathbf{0.22} & \mathbf{0.20} & \mathbf{0.32} \\ 0.20 & - & 0.30 & \mathbf{0.23} & 0.30 \\ 0.70 & \mathbf{0.49} & - & \mathbf{0.43} & \mathbf{0.53} \\ 0.60 & 0.10 & \mathbf{0.59} & - & 0.90 \\ 0.8 & \mathbf{0.49} & 1.00 & 0.30 & x \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.10 & 0.90 & \mathbf{0.40} & 0.70 \\ 0.10 & - & 0.80 & \mathbf{0.23} & 0.30 \\ 0.40 & \mathbf{0.11} & - & \mathbf{0.08} & 0.30 \\ \mathbf{0.56} & 0.10 & \mathbf{0.80} & - & 0.90 \\ 0.90 & \mathbf{0.21} & 0.10 & \mathbf{0.28} & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & \mathbf{0.30} & \mathbf{0.40} & \mathbf{0.83} & \mathbf{0.53} \\ 0.10 & - & \mathbf{0.43} & 0.90 & \mathbf{0.30} \\ 0.30 & \mathbf{0.40} & - & \mathbf{0.65} & \mathbf{0.40} \\ 0.50 & \mathbf{0.36} & 0.40 & - & \mathbf{0.35} \\ 0.30 & 0.10 & \mathbf{0.35} & \mathbf{0.60} & - \end{pmatrix}$$

As an example of illustration, the procedure to estimate the missing values in P^1 is as follows:

- The missing values that may be estimated in the initial step are:

  $$EMV_1 = \{(2,1),(2,3),(2,4),(3,1),(3,2),(3,5),(4,1),(4,3),(4,5),(5,2),(5,3),(5,4)\}$$

  We have the following incomplete fuzzy preference relation once these missing values have been estimated:

  $$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ \mathbf{0.10} & - & \mathbf{0.35} & \mathbf{0.54} & 0.20 \\ \mathbf{0.23} & \mathbf{0.34} & - & 0.30 & \mathbf{0.35} \\ \mathbf{0.58} & 0.80 & \mathbf{0.59} & - & \mathbf{0.47} \\ x & \mathbf{0.46} & \mathbf{0.49} & \mathbf{0.43} & - \end{pmatrix}$$

  For instance, the procedure to estimate p^1_{21} is:

  $$H^{11}_{21} = \emptyset \quad \Rightarrow \quad cp^1_{43} = 0$$
  $$H^{12}_{21} = \emptyset \quad \Rightarrow \quad cp^2_{43} = 0$$
  $$H^{13}_{21} = \{5\} \quad \Rightarrow \quad cp^3_{43} = cp^{53}_{43} = p_{25} - p_{15} + 0.50 = 0.10$$

  $$\mathcal{K} = 1 \quad \Rightarrow \quad cp_{43} = \frac{0 + 0.10 + 0}{1} = 0.10$$

- In the second step, we may estimate the following missing value:

  $$EMV_2 = \{(5,1)\}$$

  We obtain the following complete fuzzy preference relation once this missing value has been estimated:

  $$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ \mathbf{0.10} & - & \mathbf{0.35} & \mathbf{0.54} & 0.20 \\ \mathbf{0.23} & \mathbf{0.34} & - & 0.30 & \mathbf{0.35} \\ \mathbf{0.58} & 0.80 & \mathbf{0.59} & - & \mathbf{0.47} \\ \mathbf{0.43} & \mathbf{0.46} & \mathbf{0.49} & \mathbf{0.43} & - \end{pmatrix}$$

Once all the missing values have been estimated, using the method presented in Section 2.2, we measure the consistency degree related to each fuzzy preference relation:

$$cd^1 = 0.917 \quad cd^2 = 0.872 \quad cd^3 = 0.851 \quad cd^4 = 0.920$$

Then, the global consistency is equal to $(0.917 + 0.872 + 0.851 + 0.920)/4 = 0.890$.

Before applying our approach, it becomes informative to study the effect of the deterioration or improvement of the consistency degree related to the fuzzy preference relations when provided with an imposed level of granularity. A particular value of the level of granularity is allowed to study the impact of the given value for a given fuzzy preference relation P. Then, coming from a granular representation of P, $\boldsymbol{I}(P)$, we generate in a random manner a fuzzy preference relation and compute its corresponding consistency degree. We repeat the calculations 500 times for each value of the level of granularity. In Fig. 3, we show the related plots of the consistency degree in contrast to the given level of granularity. Furthermore, the mean of the consistency degrees are also displayed in these plots.

Theoretically, when the value of the level of granularity increases, the probability of estimating missing values so that we arrive at a more consistent fuzzy preference relation also increases. It is expected as we intend to exploit the flexibility inserted by the granularity level. However, the likelihood of producing very inconsistent preference relations also increases. Despite this, the average value of the consistency degrees presents some slight downward trend for higher values of the granularity level. Particularly, if the number of missing values is very high, the average consistency degree related to the fuzzy preference relation usually decreases for higher values of the level of granularity.

Once studied the effect of the imposed level of granularity in the deterioration or improvement of the consistency degree related to the fuzzy preference relation, we run the approach presented in Section 3.2 to optimize the estimated values assumed by the entries with missing values of the fuzzy preference relations. Taking into consideration different selected values of α, Fig. 4 displays the performance of the particle swarm optimization in relation to the values reported by the fitness function in consecutive generations. At the beginning of the optimization process (first 400 generations) we may observe the most significant improvement. After that, we may observe a slight upward trend until a clearly visible stabilization is reached in the last generations, that is, the values reported by the fitness function are constant.

Comparing with the consistency degrees obtained by the estimation procedure described in Section 2.3 (it is similar to assume a granularity level α equal to 0), our proposal achieves better results (see Fig. 4 and Table 1). As we may observe, a higher imposed level of granularity implies higher values reported by the fitness function and, therefore, the consistency degrees associated with the complete fuzzy preference relations are also higher. It is important to keep in mind that a higher level of granularity implies a higher flexibility introduced in the fuzzy preference relations, which increases the probability of completing incomplete fuzzy preference relations of higher consistency. However, this improvement is not so high as it might be expected. It is due to the fact that the missing values are estimated so that the consistency related to the complete fuzzy preference relation is higher. Anyway, the optimization of the estimated values achieves better consistency degrees than the estimation procedure described in Section 2.3 (when $\alpha = 0$).

Finally, as illustration example, the following complete fuzzy preference relations are obtained when the estimated values are optimized with $\alpha = 1$ (the estimated values are in bold):

$$P^1 = \begin{pmatrix} - & 0.20 & 0.40 & 0.60 & 0.60 \\ \mathbf{0.37} & - & \mathbf{0.41} & \mathbf{0.34} & 0.20 \\ \mathbf{0.47} & \mathbf{0.50} & - & 0.30 & \mathbf{0.42} \\ \mathbf{0.71} & 0.80 & \mathbf{0.66} & - & \mathbf{0.59} \\ \mathbf{0.57} & \mathbf{0.54} & \mathbf{0.53} & \mathbf{0.43} & - \end{pmatrix}$$

Table 1
Results achieved by different values of α.

	cd^1	cd^2	cd^3	cd^4	f
$\alpha = 0$	0.917	0.872	0.851	0.920	0.890
$\alpha = 0.5$	0.946	0.897	0.866	0.948	0.914
$\alpha = 1$	0.946	0.898	0.866	0.949	0.915
$\alpha = 1.5$	0.947	0.898	0.866	0.950	0.915
$\alpha = 2$	0.948	0.899	0.866	0.951	0.916

$$P^2 = \begin{pmatrix} - & \mathbf{0.34} & \mathbf{0.52} & \mathbf{0.22} & \mathbf{0.41} \\ 0.20 & - & 0.30 & \mathbf{0.10} & 0.30 \\ 0.70 & \mathbf{0.45} & - & \mathbf{0.28} & \mathbf{0.49} \\ 0.60 & 0.10 & \mathbf{0.73} & - & 0.90 \\ 0.8 & \mathbf{0.56} & 1.00 & 0.30 & - \end{pmatrix}$$

$$P^3 = \begin{pmatrix} - & 0.10 & 0.90 & \mathbf{0.42} & 0.70 \\ 0.10 & - & 0.80 & \mathbf{0.36} & 0.30 \\ 0.40 & \mathbf{0.19} & - & \mathbf{0.25} & 0.30 \\ \mathbf{0.59} & 0.10 & \mathbf{0.83} & - & 0.90 \\ 0.90 & \mathbf{0.21} & 0.10 & \mathbf{0.37} & - \end{pmatrix}$$

$$P^4 = \begin{pmatrix} - & \mathbf{0.43} & \mathbf{0.66} & \mathbf{0.82} & \mathbf{0.71} \\ 0.10 & - & \mathbf{0.61} & 0.90 & \mathbf{0.63} \\ 0.30 & \mathbf{0.31} & - & \mathbf{0.60} & \mathbf{0.51} \\ 0.50 & \mathbf{0.27} & 0.40 & - & \mathbf{0.39} \\ 0.30 & 0.10 & \mathbf{0.42} & \mathbf{0.53} & - \end{pmatrix}$$

In summary, it may be concluded that an incomplete fuzzy preference relation may be completed so that the consistency associated with it is higher with the usage of the approach presented in this study. It speaks to the information granularity plays a notable role in the improvement of consistency.

4.2. Second study

In the second study, we suppose the following incomplete fuzzy preference relation:

$$P = \begin{pmatrix} - & 0.30 & 0.60 & 0.70 & x \\ 0.80 & - & x & x & x \\ x & x & - & 0.40 & x \\ 0.20 & 0.60 & x & - & x \\ x & x & x & x & - \end{pmatrix}$$

The estimation procedure presented in Section 2.3 may estimate all the missing values encountered in a fuzzy preference relation if a set of $n - 1$ non-leading diagonal preference values is known, where each one of the options is compared at least once [22]. Therefore, in this case, it cannot estimate all the missing values as the option o_5 is never compared.

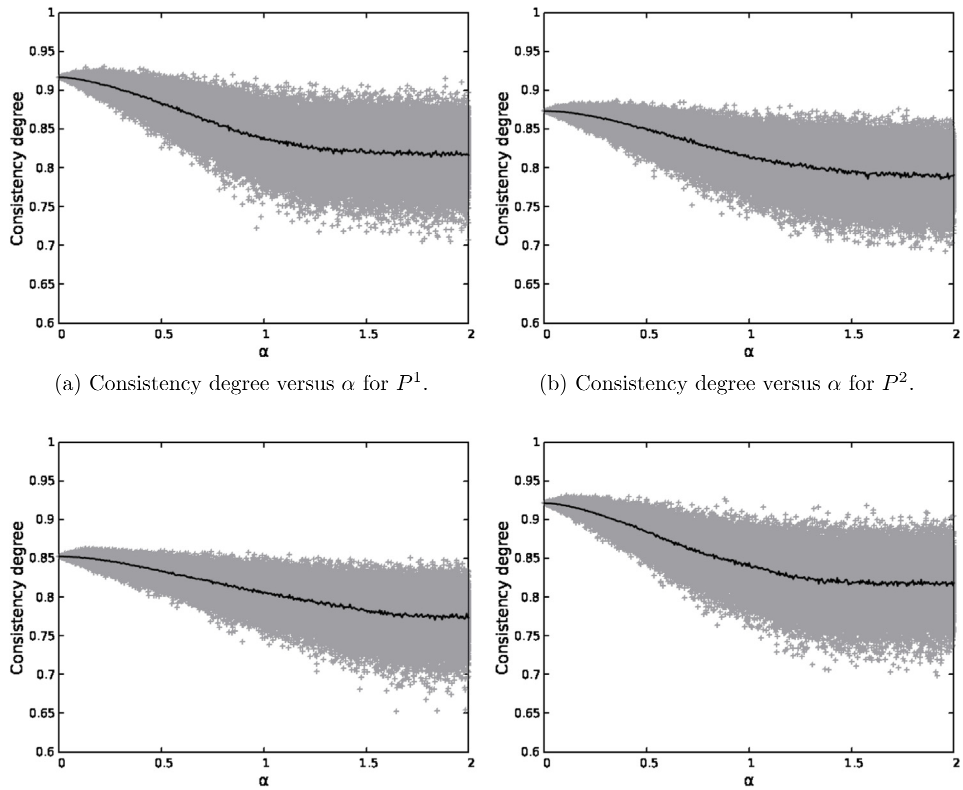
However, we may apply our approach by assuming that the missing values that may be not estimated using the estimation procedure presented in Section 2.3 can assume any value in the unit interval. Therefore, in such a case, the level of granularity assumed is equal to 2. Fig. 5 displays the progression of the values reported by the fitness function.

In this case, the complete fuzzy preference relation obtained is the following (the estimated values are in bold):

$$P = \begin{pmatrix} - & 0.30 & 0.60 & 0.70 & \mathbf{0.49} \\ 0.80 & - & \mathbf{0.87} & \mathbf{0.97} & \mathbf{0.68} \\ \mathbf{0.30} & \mathbf{0.21} & - & \mathbf{0.40} & \mathbf{0.31} \\ 0.20 & 0.60 & \mathbf{0.56} & - & \mathbf{0.39} \\ \mathbf{0.52} & \mathbf{0.41} & \mathbf{0.70} & \mathbf{0.67} & - \end{pmatrix}$$

being the consistency degree related to it equal to 0.951.

In summary, in addition to improve the consistency degree obtained by the estimation procedure presented in Section 2.3, the proposed approach may be also applied in situations in which



(a) Consistency degree versus α for P^1.

(b) Consistency degree versus α for P^2.

(c) Consistency degree versus α for P^3.

(d) Consistency degree versus α for P^4.

Fig. 3. Plots of consistency degrees versus α.

Table 2
Consistency degrees achieved by [22] and the proposed approach.

	$o = 5$	$o = 10$	$o = 15$	$o = 20$
$m = 5$	0.811 **0.853**	0.723 **0.747**	0.827 **0.840**	0.888 **0.910**
$m = 10$	0.743 **0.776**	0.655 **0.682**	0.677 **0.698**	0.777 **0.801**
$m = 15$	0.645 **0.688**	0.803 **0.844**	0.901 **0.923**	0.798 **0.823**
$m = 20$	0.754 **0.788**	0.771 **0.803**	0.697 **0.727**	0.866 **0.891**

the above estimation procedure does not work. However, in this case, it estimates preference degree for options that have not been compared at least once and, therefore, even though the preference degrees are estimated so that the consistency level associated with the fuzzy preference relation is as high as possible, the estimated values should be presented to the individual in order that she/he accepts them.

4.3. Third study

In this third study, we test the performance of the proposed approach in different scenarios in which we assume a higher number of individuals (m) and options (o). To do so, we randomly generate incomplete fuzzy preference relations and apply the approach presented in [22] and the proposed approach to complete them.

Table 2 shows the results achieved by the approach presented in [22] (in normal font) and the results achieved by the proposed approach (in bold) in terms of the fitness function f. In the above examples, we have observed that the proposed approach achieves better results when the maximum level of granularity is assumed. Therefore, in this third study, we set $\alpha = 2$. It can be seen that the proposed approach obtains complete fuzzy preference relations with higher consistency degrees.

5. Concluding remarks

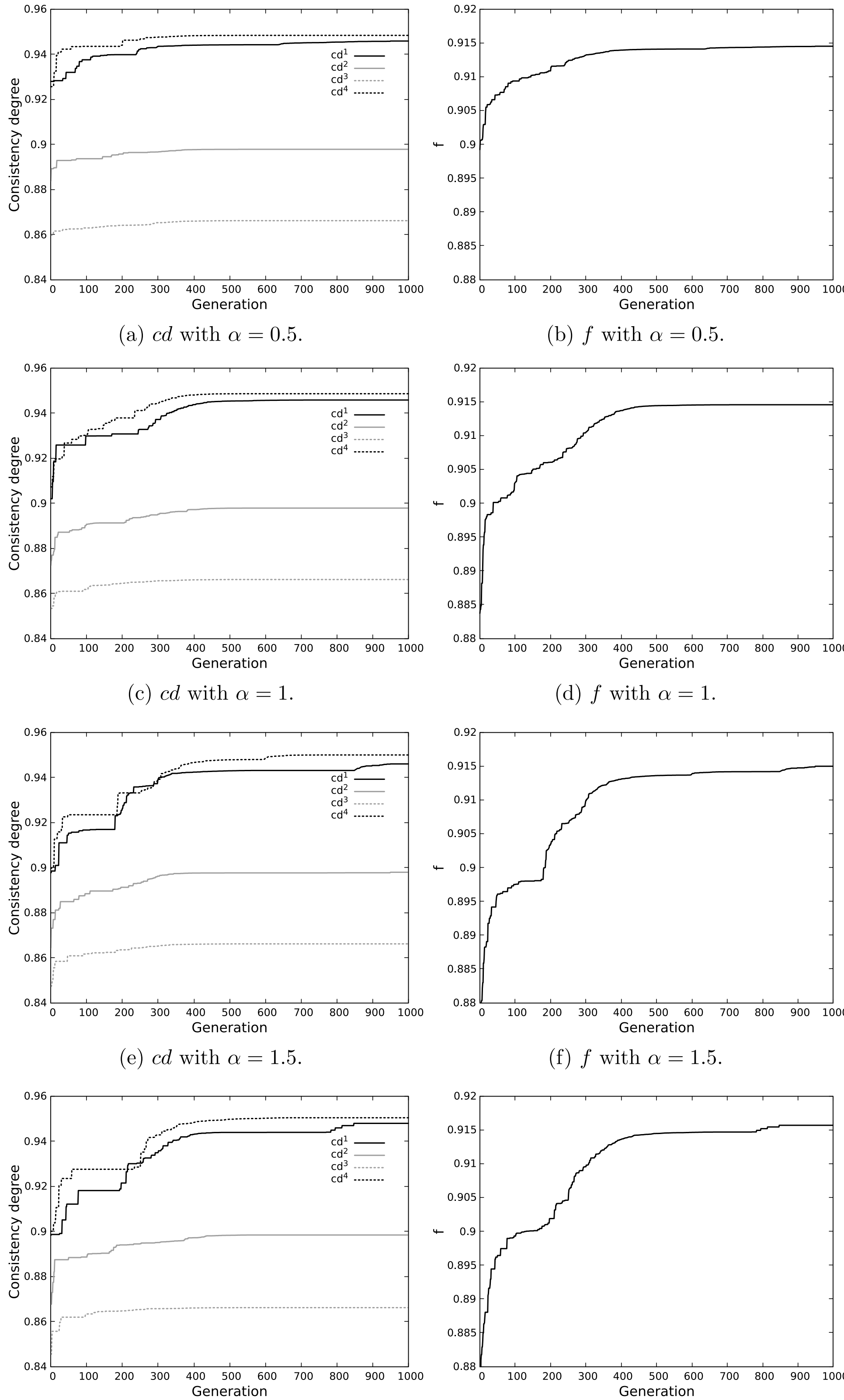
This study has formulated, motivated, and solved the problem of estimating missing values of incomplete fuzzy preference relations so that the consistency degree related to the complete fuzzy preference relations obtained are as higher as possible.

This investigation is in line of a general position aligned with the principles of information granularity and the very nature of the resulting information granules. By starting with a collection of incomplete fuzzy preference relations, we have presented a comprehensive algorithm framework that comes up with granular preference relations (in particular, intervals) to estimate the missing values. We have emphasized the motivation and need behind engaging information granules so that the missing values have been estimated to obtain fuzzy preference relations of higher consistency.

We have also shown that the particle swarm optimization algorithm serves as an appropriate optimization framework. However, we should note that while this framework maximizes the values reported by the fitness function, it does not guarantee an optimal result, rather than we may refer to it as the best solution that is produced by the particle swarm optimization framework.

We conclude with some suggestions for future studies:

- In this study, we have shown how to elevate the estimation procedure presented in [22] to its granular form. However, the proposed approach may also be applied to any other numeric estimation procedure [44–48].



(a) cd with $\alpha = 0.5$.
(b) f with $\alpha = 0.5$.
(c) cd with $\alpha = 1$.
(d) f with $\alpha = 1$.
(e) cd with $\alpha = 1.5$.
(f) f with $\alpha = 1.5$.
(g) cd with $\alpha = 2$.
(h) f with $\alpha = 2$.

Fig. 4. Plots of the consistency degrees and f in successive generations.

- The allocation of the information granularity was expressed in terms of a uniform distribution, in which all numeric estimated values were treated similarly and became substituted by intervals of the same length that were distributed symmetrically around the estimated values. There is, however, a wealth of possibilities to investigate when it comes to the

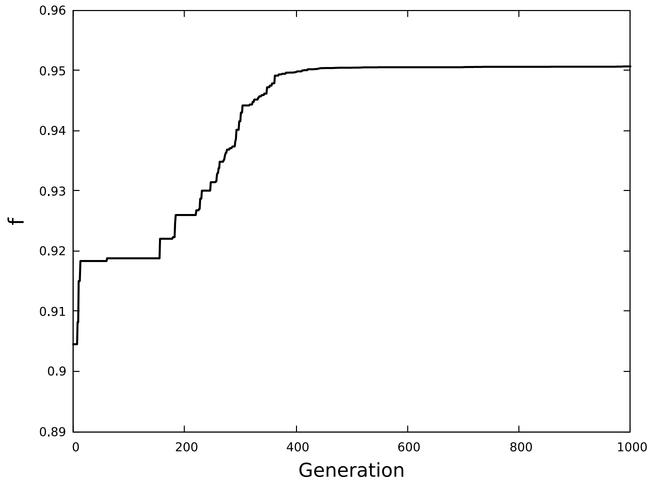


Fig. 5. Values reported by f in successive generations.

allocation of the available information granularity: (i) uniform allocation of information granularity with asymmetric position of intervals around the estimated values, (ii) non-uniform allocation of information granularity with symmetrically distributed intervals, and (iii) non-uniform allocation of information granularity with asymmetrically distributed intervals. Furthermore, to assess the relative performance of the above approaches, an interesting reference point is to consider a random allocation of the information granularity. It helps quantify how the optimized and meticulously planned process of allocation of information granularity is better than a simply random allocation process.

- We focused on the formalization of information granules as intervals for the conciseness and clarity of the presentation. However, the underlying conceptual framework is also appropriate to cope with other formal realizations of information granules as, for instance, Pythagorean fuzzy sets [49,50].

Declaration of competing interest

No author associated with this paper has disclosed any potential or pertinent conflicts which may be perceived to have impending conflict with this work. For full disclosure statements refer to https://doi.org/10.1016/j.asoc.2019.105930.

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